BRIEF COURSE

IN

ANALYTIC GEOMETRY

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PREFACE

In their "Elementary Course in Analytic Geometry" the authors of this book endeavored to prepare a text suitable for the general student, as well as for the student who desires a definite scientific training, including preparation for advanced mathematical study. The success of the "Elementary Course" has been demonstrated by its adoption and continued use in many of the leading colleges and scientific schools.

Since the publication of that text, however, there has been a growing desire for a somewhat briefer book preserving the same rigor of proofs and careful analysis, but omitting or shortening the less important details. The "Brief Course in Analytic Geometry" is prepared to meet this demand.

The chief features of the "Elementary Course" which have won such universal approval have been included. These are:

(1) An extended introduction to the method of the analytic geometry, — showing the value of the interpretation of the negative number by means of coördinates, the relation of the locus to the equation, and the use of the equation to find and demonstrate properties of the locus.

(2) Introduction of the demonstration of general theorems by numerical examples, in order to make clear the *method* used before bringing in the added difficulties of literal notation and general proofs.

(3) The use of some intrinsic properties of curves, to emphasize further the fact that the coördinates are a tool, an intermediate device, not essential to the properties which are studied by their aid.

(4) Rigorous proof of all the theorems within the scope of the book.

(5) An abundance of carefully graded numerical exercises.

In preparing the "Brief Course" the senior author participated actively in the planning of the scope and general arrangement of the work, but the junior author is responsible for the working out of most of the details.

Cordial acknowledgment is due to Dr. A. B. Turner, College of the City of New York, who read both the manuscript and the proofs, and to Mr. G. M. Green, who prepared the answers

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ANALYTIC GEOMETRY

PART I

CHAPTER I

INTRODUCTION*

ALGEBRAIC AND TRIGONOMETRIC CONCEPTIONS

1. Number. When one quantity is measured by another quantity of the same kind chosen as a unit, the result is expressed as a number, **positive** (+) or **negative** (−). If the algebraic operations, addition, etc., are performed upon given numbers, the result is in each case a number. A number is **imaginary** if it involves in any way an indicated even root of a negative number; otherwise it is **real.**

The **absolute value** of a number is its value irrespective of its sign.

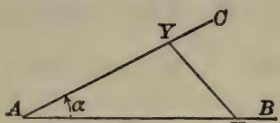

FIG. 1.

2. Constants and variables. If AB and AC are two given straight lines making an angle α at the point A, and if any two points X and Y, on these lines, respectively, are joined by a straight line, then

Area of triangle $AXY = \frac{1}{2} \cdot AX \cdot AY \cdot \sin\alpha$,

i.e., $$\Delta = \tfrac{1}{2} \cdot x \cdot y \cdot \sin\alpha,$$

* This introduction is in the nature of a review, which may be omitted, or made more complete, at the discretion of the teacher.

where x is the length of AX, y is the length of AY, and Δ is the area of the triangle.

If now the points X and Y are moved along the lines AB and AC in any way whatever, then Δ, x, and y will each pass through a series of different values, — they are **variable numbers** or **variables**; while $\frac{1}{2}$ and $\sin\alpha$ will **remain** unchanged, — they are **constant numbers** or **constants**.

It is to be remarked that $\frac{1}{2}$ has the **same value** wherever it occurs, — it is an *absolute* constant; while α, though constant for this series of triangles, may have a different constant value for another series of triangles, — it is an *arbitrary* constant.

Because x and y may separately take any values whatever, they are *independent* variables; while Δ, whose value depends upon the values of x and y, is a *dependent* variable.

The illustrations just given may serve to give a clearer conception of the following more formal definitions.

An **absolute constant** is a number that has the same value wherever it occurs; such are the numbers 2, 7, $\frac{3}{5}$, $6^{\frac{2}{3}}$, π, e (where $\pi = 3.14159265 \cdots$, approximately $\frac{22}{7}$, the ratio of the circumference of a circle to its diameter;

and $$e = 2.71828182 \cdots = 1 + \frac{1}{1} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \cdots,$$

approximately $\frac{19}{7}$, the base of the Naperian system of logarithms).

An **arbitrary constant** is a number that retains the same value throughout the investigation of a given problem, but may have a different fixed value in another problem.

An **independent variable** is a number that may take any value whatever within limits prescribed by the conditions of the problem under consideration.

A **dependent variable** is a number that depends for its

value upon the values assumed by one or more independent variables.

A variable that becomes greater than any assigned number, however great, is an **infinite number**; a variable that becomes and remains smaller (numerically, not merely algebraically *less*) than any assigned number, however small, is an **infinitesimal number**. All other numbers are **finite**.

3. Functions. A number so related to one or more other numbers that it depends upon these for its value, is a **function** of these other numbers. *E.g.*, the circumference and the area of a circle are functions of its radius; the distance traveled by a railway train is a function of its time and rate; if $y = 3x^2 + 5x - 8$, then y is a function of x.

4. Identity, equation, and root. If two functions involving the same variables are equal to each other for *all* values of those variables, they are *identically* equal. Such an equality is expressed by writing the sign $\equiv$ between the two functions, and the expression so formed is an **identity**. If, on the other hand, the two functions are equal to each other only for particular values of the variables, the equality is expressed by writing the sign $=$ between the two functions, and the expression so formed is an **equation**. The particular values for which the two functions are equal, *i.e.*, those values of the variables which *satisfy* the equation, are the **roots** of the equation.

E.g., $(x+y)^2 \equiv x^2 + 2xy + y^2$, $(x+a)(x-a) + a^2 \equiv x^2$,

and $x + \dfrac{3}{x-1} \equiv \dfrac{x^2 - x + 3}{x-1}$ are identities;

while $3x^2 - 10x + 2 = 2x^2 - 4x - 6,$

is an equation. The roots of this equation are the numbers 2 and 4.

Special attention is called to the fact that an equation always imposes a *condition*: an identity does not.

E.g., $x^2-6x+8=0$ if, and only if, $x=2$ or $x=4$.

So also the equation $ax+by+c=0$ imposes the condition that x shall be equal to

$$\frac{-by-c}{a}.$$

5. Notation. In general, absolute constants are represented by the Arabic numerals, while arbitrary constants and variables are represented by letters. A few absolute constants are, however, by general consent, represented by letters; examples of such constants are π and e. [Art. 2.] Variables are usually represented by the last letters of the alphabet, such as u, v, w, x, y, z; while the first letters, $a, b, c, \cdots$ are reserved to represent constants, especially arbitrary constants.

Particular fixed values from among those that a variable may assume are sometimes in question; *e.g.*, the values, $x=2$ and $x=-1$, for which the function x^2-x-2 vanishes; such values may conveniently be denoted by affixing a subscript to the letter representing the variable. Thus $x_1, x_2, x_3, \cdots$ will be used to denote particular values of the variable x.

Similarly, variables which enter a problem in analogous ways are usually denoted by a single letter having accents attached to it; thus $x', x'', x''', \cdots$ denote variables that are similarly involved in a given problem.

The word "function" is usually represented by a single letter, such as $f, F, \phi, \psi, \cdots$; thus $y=\phi(x)$ means that y is a function of the independent variable x, and is read "y equals the ϕ-function of x"; so also $z=F(u, v, x)$ means that z is a function of the independent variables, u, v, and x, and it is read, "z equals the F-function of u, v, and x."

Each of the two equations, $y = 3x^2 - 4x + 10$ and $y = \phi(x)$, asserts that y is a function of x; but while the former tells precisely *how* y depends upon x, the latter merely asserts that *there is* such a dependence, without giving any information concerning the form of that dependence.

> If, however, the form of ϕ in a given problem is defined by the equation
>
> $$\phi(x) = \frac{3x^5 - x^4 + 5}{2x + 1},$$
>
> then, *in the same problem*
>
> $$\phi(v) = \frac{3v^5 - v^4 + 5}{2v + 1}, \quad \phi(1) = \frac{7}{3}, \text{ and } \phi(0) = 5.$$

6. The quadratic equation. Its solution. The most general equation of the second degree, in one unknown number, may be written in the form

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c are known numbers. Its roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

Their nature depends upon the number $b^2 - 4ac$, called the **discriminant** of the quadratic equation; there are three cases to be considered, viz.:

$$\left.\begin{array}{lll} \textit{if} & b^2 - 4ac > 0, & \textit{then the roots are real and unequal,} \\ \textit{if} & b^2 - 4ac = 0, & \textit{then the roots are real and equal,} \\ \textit{if} & b^2 - 4ac < 0, & \textit{then the roots are imaginary.} \end{array}\right\} \tag{3}$$

EXERCISES

1. Show which of the following equalities are identities:

(1) $x^2-4x+4=0$; (4) $(p+q)^3=p^3+q^3+3pq(p+q)$;

(2) $(s+t)(s-t)=s^2-t^2$; (5) $x^2+5x+6=(x+3)(x+2)$.

(3) $\dfrac{\alpha^3+\beta^3}{\alpha+\beta}=\alpha^2-\alpha\beta+\beta^2$;

2. Determine, without solving the equation, the nature of the roots of

$$3x^2+8x+1=0.$$

SOLUTION. Since $b^2-4ac=64-12=52$, *i.e.*, is positive, therefore the roots are real and unequal; again since a, b, and c are all positive, therefore both roots are negative [cf. (2), Art. 6].

3. Without solving the equation, determine the character of the roots of

$$8x^2-3x+1=0.$$

4. Given the equation $x^2-3x-m(x+2x^2+4)=5x^2+3$. Find the roots. For what values of m are these roots equal?

5. Determine, without solving, the character of the roots of the equations:

(1) $5z^2-2z+5=0$; (2) $x^2+7=0$; (3) $3t^2-t=19$.

6. Determine the values of m for which the following equations shall have equal roots:

(1) $x^2-2x(1+3m)+7(3+2m)=0$;

(2) $mx^2+2x^2-2m=3mx-9x+10$;

(3) $4x^2+(1+m)x+1=0$; (4) $x^2+(6x+m)^2=a^2$.

7. If in the equation $2ax(ax+nc)+(n^2-2)c^2=0$, x is real, show that n is not greater, in absolute value, than 2

8. If x is real in the equation $\dfrac{x}{x^2-5x+9}=a$, show that a is not greater than 1, nor less than $-\frac{1}{11}$.

9. For what values of c will the following equations have equal roots?

(1) $3x^2+4x+c=0$; (2) $(mx+c)^2=4lx$;

(3) $4x^2+9(2x+c)^2=36$.

10. Solve the equations in examples 2, 3, and 5.

11. Solve the equations:

(1) $x^4-25x^2=-144$; (2) $\dfrac{3x-2}{x-2}-\dfrac{2x+1}{x+2}+\dfrac{12}{x^2-4}=0$.

7. **Zero and infinite roots.** In equations (2) of Art. 6, x_1 and x_2, *i.e.*, the roots of $ax^2+bx+c=0$, were given; and it is seen that

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}=\frac{2c}{-b-\sqrt{b^2-4ac}}, \qquad (1)$$

and that

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{2c}{-b+\sqrt{b^2-4ac}}. \qquad (2)$$

Equations (1) and (2) show that:

(1) If a and b remain unchanged while c grows smaller (numerically), then x_1 grows smaller and x_2 grows larger: the roots of $ax^2+bx+0=0$ are $x_1=0$, $x_2=-\dfrac{b}{a}$.

(2) If a remains unchanged while c and b grow smaller, then both x_1 and x_2 grow smaller: the roots of $ax^2+0\cdot x+0=0$ are $x_1=0$, $x_2=0$.

(3) If b and c remain unchanged while a grows smaller, then one root, x_2, grows larger, indefinitely; the roots of $0 \cdot x^2 + bx + c = 0$ are $x_1 = -\frac{c}{b}$, $x_2 = \infty$.

(4) If c remains unchanged while a and b become smaller, then both x_1 and x_2 grow larger, indefinitely.

(5) If a and c remain unchanged while b grows smaller, then the roots approach numerical equality: the roots of $ax^2 + 0 \cdot x + c = 0$ are $x_1 = \sqrt{-\frac{c}{a}}$, $x_2 = -\sqrt{-\frac{c}{a}}$.

8. Properties of the quadratic equation. If the roots of a quadratic equation are not themselves needed, but only their sum or product is desired, these may be obtained directly from the given equation by inspection; for by adding, and again by multiplying, the two roots of

$$ax^2 + bx + c = 0, \tag{1}$$

are obtained the relations

$$x_1 + x_2 = -\frac{b}{a}, \qquad x_1 x_2 = \frac{c}{a}. \tag{2}$$

E.g., the half sum of the roots of the equation

$$m^2x^2 + 2(bm - 2\,l)x + b^2 = 0$$

is

$$\frac{x_1 + x_2}{2} = -\frac{2(bm - 2\,l)}{2\,m^2} = \frac{2\,l - bm}{m^2}.$$

Again, if x_1 and x_2 are the roots of the equation

$$x^2 + px + q = 0,$$

then $x - x_1$ and $x - x_2$ are the factors of its first member.

Also conversely: if a quadratic function can be separated into two factors of the first degree, then the roots can be immediately written by inspection.

E.g., the roots of $(2\,x - 3)(x + 5) = 0$ are $\frac{3}{2}$ and -5.

9. The quadratic equation involving two unknowns. One equation involving two unknown numbers cannot be solved uniquely for the values of those numbers which satisfy the equation; but if there is assigned to either of those numbers a definite value, then at least one definite and corresponding value can be found for the other, so that, this *pair* of values being substituted for the unknown numbers, the equation will be satisfied. In this way an infinite number of pairs of values, that will satisfy the equation, may be found.

If, however, the equation is *homogeneous* in the two unknowns, *i.e.*, of the form

$$ax^2 + bxy + cy^2 = 0,$$

then the ratio $x : y$ may be regarded as a single number, and the equation has properties precisely like those discussed in Arts. 6, 7, and 8.

To solve a system consisting of two or more independent simultaneous equations, involving as many unknown elements, it is necessary to combine the equations so as to eliminate all but one of the unknown elements, then to solve the resulting equation for that one, and, by means of the roots thus obtained, find the entire system of roots.

EXERCISES

1. Given the equation $x^2 + 3x - 4 + m(3x^2 - 4) - 2mx^2 = 0$, find the sum of the roots; the product of the roots; also the factors of the first member.

2. Factor the following expressions:

(1) $x^2 - 5x + 4$; (3) $mx^2 - 3x + c$; (5) $3w^{\frac{5}{3}} - 94w^{\frac{5}{6}} - 64$;

(2) $x^2 + 2x - 8$; (4) $ax^2 + bxy + cy^2$; (6) $11 - 27y - 18y^2$.

3. Without first solving the equation

$$x^2 - 3x - m(x + 2x^2 + 4) = 5x^2 + 3,$$

find the sum, and the product, of its roots. For what value of m are its roots equal? For what value of m does one root become infinitely large? If all the terms are transposed to one member, what are the factors of that member?

4. Without first solving, determine the nature of the roots of the equation $(m-2)(\log x)^2-(2m+3)\log x-4m=0$. [Regard $\log x$ as the unknown element.]

For what values of m are the roots equal? real? one infinitely great? one zero? Find the factors of the first member of the equation.

5. Find five pairs of numbers that satisfy the equation:

(1) $x+3y-7=0$; (3) $y^2=16x$;

(2) $x^2+y^2=4$; (4) $3x+6xy-8y^2+3x^2=0$.

6. Without solving, determine the nature of the roots of the equation:
$$9x^2+12xy+4y^2=0, \quad 3u^2-uv+19v^2=0.$$

7. Solve the following pairs of simultaneous equations:

(1) $3x-5y+2=0$, and $2x+7y-4=0$;

(2) $5y+2z+3=0$, and $7y+4z+2=0$;

(3) $y-3x+c=0$, and $y^2=9x$;

(4) $x^2+y^2=5$, and $y^2=6x$;

(5) $b^2x^2+a^2y^2=a^2b^2$, and $y=ax+b$;

(6) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$, and $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$.

8. Determine those values of b for which each of the following pairs of equations will be satisfied by two equal values of y:

(1) $\{x^2+y^2=a^2,\ y=6x+b\}$; (2) $\{y=mx+b,\ y^2=4x\}$;

(3) $\{3y+2x=b,\ 6x^2+y^2=12\}$

9. Determine, for the pairs of equations in Ex. 8, those values of b which will give equal values of x.

10. Directed lines. Angles. A line is said to be **directed** when a distinction is made between the segment from any point A of the line to another point B, and the opposite segment from B to A. One of these directions is chosen as positive, or $+$, and the opposite direction is then negative or $-$. The portion of the line on one side of a point is a *half line*, or *ray*. An **angle** is the figure formed by two half lines from a point. If the lines be directed, the angle is measured by the amount of rotation about their point of intersection necessary to bring the positive end of the initial side into coincidence with the positive end of the terminal side. The point in which the lines meet is called the **vertex** of the angle. The angle is *positive*, or $+$, if the rotation from the initial to the terminal side is in *counter-clockwise* direction; the angle is *negative*, or $-$, if the rotation is *clockwise*.

The angle formed by two directed straight lines in space, which do not meet, is equal to the angle between two rays which meet and are respectively parallel to the given lines.

The angle from the directed line a to the directed line b may be denoted as $\angle(ab)$.

For the measurement of angles there are two absolute units:

(1) *The angular magnitude about a point in a plane, i.e.*, a complete revolution. An angle of one fourth of a complete revolution is called a **right angle**; $\frac{1}{90}$ of a right angle is a **degree** (1°); $\frac{1}{60}$ of a degree is a **minute** ($1'$); and $\frac{1}{60}$ of a minute is a **second** ($1''$).

(2) *The angle whose subtending circular arc is equal in length to the radius of that arc*; this angle is called a **radian** $\{1^{(r)}\}$; it is independent of the length of the radius.

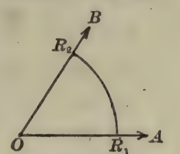

FIG. 2.

These units have the following relation:

$$\pi^{(r)} = 180^\circ;$$

Therefore, $1^{(r)} = \dfrac{180°}{\pi} = 57°\ 17'\ 44.8''$, approximately.

A right angle is $90°$ or $\left(\dfrac{\pi}{2}\right)^{(r)}$.

11. Trigonometric ratios. If from any point P in the terminal side of an angle θ, at a distance r from the vertex, a perpendicular MP is drawn to the initial side meeting it in

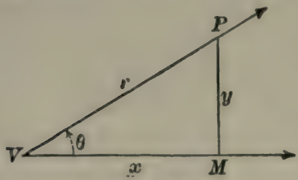

Fig. 3 *a*.

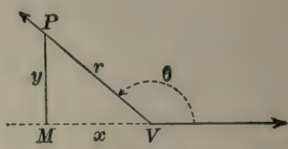

Fig. 3 *b*.

M, and if MP be represented by y and VM by x, then, by general agreement, y is $+$ if MP makes a positive right angle with the initial line, and $-$ if this right angle is negative; similarly, x is $+$ if VM extends in the positive direction of the initial line, and $-$ if it extends in the opposite direction.

The $\triangle\, VMP$ is the **reference triangle** for $\angle\,\theta$. Its three sides form six ratios, known as the **trigonometric ratios** or **functions** of the angle θ, and are named as follows:

sine $\theta = \dfrac{y}{r}$, tangent $\theta = \dfrac{y}{x}$, secant $\theta = \dfrac{r}{x}$,

cosine $\theta = \dfrac{x}{r}$, cotangent $\theta = \dfrac{x}{y}$, cosecant $\theta = \dfrac{r}{y}$.

These functions are not all independent, but are connected by the following relations:

Functions of an angle.

(1) $\sin\theta \cdot \csc\theta = 1$,

(2) $\cos\theta \cdot \sec\theta = 1$,

(3) $\tan\theta \cdot \cot\theta = 1$,

(4) $\tan\theta = \sin\theta : \cos\theta$,

(5) $\cot\theta = \cos\theta : \sin\theta$,

(6) $\sin^2\theta + \cos^2\theta = 1$,

(7) $\tan^2\theta + 1 = \sec^2\theta$,

(8) $\cot^2\theta + 1 = \csc^2\theta$.

Functions of related angles. Based upon the definitions of the trigonometric functions the following relations are readily established for any plane angle, θ.

(9) function of $(180° \pm \theta)$, or $(360° \pm \theta) = \pm$ same function of θ,

(10) function of $(90° \pm \theta)$, or $(270° \pm \theta) = \pm$ co-function of θ.

The proper sign in each case is shown in the following table:

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin\theta$ $\csc\theta$	+	+	−	−
$\cos\theta$ $\sec\theta$	+	−	−	+
$\tan\theta$ $\cot\theta$	+	−	+	−

12. Other important formulas. If θ_1 and θ_2 are any two plane angles, then

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2,$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2,$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}.$$

If θ is any plane angle, then

$$\sin 2\theta = 2 \sin\theta \cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1,$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta},$$

$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1-\cos\theta)},$$

$$\cos\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1+\cos\theta)},$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

If a, b, and c are the sides of a triangle, directed continuously as in the figure, from A to B, B to C, C to A, and lying respectively opposite the interior angles A, B, and C; and if Δ is the area of this triangle; then

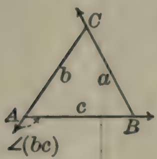

FIG. 4.

$$\Delta = \tfrac{1}{2}\, bc \sin A.$$

$$a^2 = b^2 + c^2 - 2\, bc \cos A.$$

EXERCISES

1. Express in radians the angles:

15°; 60°; 135°; $-252°$; $\frac{5}{4}$ rt. angle; 10° 10′ 10″; 88° 2′; $(3\pi)°$.

2. Express in degrees, minutes, and seconds the angles:

$\left(\frac{\pi}{4}\right)^{(r)}$; $\left(\frac{3\pi}{5}\right)^{(r)}$; $\left(\frac{1}{4}\right)^{(r)}$; $\left(\frac{2}{9}\right)^{(r)}$; $\frac{7}{10}$ of a revolution; $\frac{5}{4}$ rt. angle.

3. Find the values of the other trigonometric functions, given:

(1) $\tan\theta = 3$; (2) $\sec x = -\sqrt{2}$; (3) $\cos\phi = \frac{1}{\sqrt{3}}$;

(4) $\sin t = \frac{1}{4}$; (5) $\cot\psi = \frac{1}{7}$; (6) $\csc u = -2$.

SOLUTION OF (1). Construct a right triangle ABC with the sides $AB = 1$ and $BC = 3$, then $\angle BAC$ is an angle whose tangent is 3. Therefore

FIG. 5.

$AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{10}$, and the other functions of the angle BAC are at once seen to be:

$$\sin\theta = \frac{3}{\sqrt{10}},\quad \cos\theta = \frac{1}{\sqrt{10}},\quad \csc\theta = \frac{\sqrt{10}}{3},$$

$$\sec\theta = \sqrt{10},\quad \text{and}\quad \cot\theta = \tfrac{1}{3}.$$

4. By means of a right triangle, with appropriate acute angles, find the numerical values of the trigonometric ratios of the following angles: 30°; 45°; 60°; 90°; 135°; and −45°.

5. Express the following functions in terms of functions of positive angles less than 90°:

$$\tan 3500°;\ -\csc 290°;\ \sin(-369°);\ -\cos\frac{11\pi}{5};$$
$$\cot(-1215°).$$

6. Solve the following equations:

(1) $\sin\theta = -\cos 210°$; (2) $\cos\theta = \sin 2\theta$; (3) $\dfrac{\cos x}{\sin x \cot^2 x} = \sqrt{3}$;
and (4) $(\sec^2 x - 1)(\csc^2 x + 1) = \frac{5}{3}$.

7. In the following identities transform the first member into the second:

(1) $\dfrac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} \equiv \dfrac{2}{\csc^2\theta} - 1$; (2) $\dfrac{\sec x + \csc x}{\sec x - \csc x} \equiv \dfrac{1 + \cot x}{1 - \cot x}$;

(3) $\csc x(\sec x - 1) - \cot x(1 - \cos x) \equiv \tan x - \sin x$;

(4) $(2r\sin\alpha\cos\alpha)^2 + r^2(\cos^2\alpha - \sin^2\alpha)^2 \equiv r^2$;

(5) $(\cos a\cos b + \sin a\sin b)^2 + (\sin a\cos b - \cos a\sin b)^2 \equiv 1$;

(6) $(r\cos\phi)^2 + (r\sin\phi\cos\theta)^2 + (r\sin\phi\sin\theta)^2 \equiv r^2$.

13. Orthogonal projection. The orthogonal projection* of a point upon a line is the foot of the perpendicular from the

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* Hereafter, unless otherwise stated, *projection* will be understood to mean *orthogonal projection*.

point to the line. In the figure, M is the projection of P upon AB. The projection of a segment PQ of a line upon another

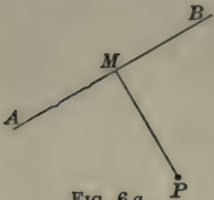

Fig. 6 a.

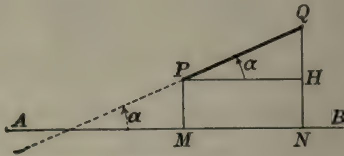

Fig. 6 b.

line AB, is that part of the second line extending from the projection of the initial point of the segment to the projection of the terminal point of the segment. Thus MN is the projection of PQ upon AB, and NM is the projection of QP upon AB.

From the definition,

$$\frac{MN}{PQ}=\frac{PH}{PQ}=\cos\alpha,$$

$$\therefore MN=PQ\cdot\cos\alpha;$$

i.e., *the projection of a segment of a line upon another line is equal to the product of its length by the cosine of the angle which it makes with that other line.*

A line made up of parts PQ, QR, RS, $\cdots$ (Fig. 7 a, 7 b), which are straight lines having different directions, is a **broken line**;

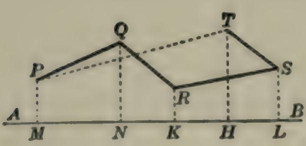

Fig. 7 a.

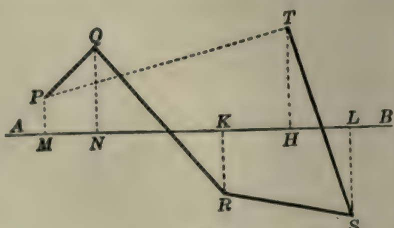

Fig. 7 b.

and the projection of a broken line* upon any line is the algebraic sum of the projections of its parts upon the same line. Thus the projection of $PQRST$ upon AB is the projection of $PQ +$ the projection of $QR + \cdots$, upon AB; *i.e.*,

$$\text{proj. } PQRST \text{ upon } AB = MN + NK + KL + LH = MH;$$

but MH is the projection of the straight line PT which joins the first initial to last terminal point of the broken line.

Similarly for any broken line.

The following theorems may therefore be stated:

(1) *The projection of a segment of a line upon any straight line in space equals the product of its length by the cosine of the angle between the two lines.*

(2) *The projections of two parallel segments of equal length upon any given line in space are equal in absolute value.*

(3) *The projection of any broken line in space upon any straight line equals the projection, upon the same line, of the straight line which joins the extremities of the broken line.*

(4) *The projection of the perimeter of any closed polygon upon any given line is zero.*

EXERCISES

1. Two lines of lengths 3 and 7 respectively meet at an angle $\frac{\pi}{3}$; find the projection of each upon the other.

2. The center of an equilateral triangle, of side 5, is joined by a straight line to a vertex; find the projection of this joining line upon each side of the triangle.

3. A rectangle has its sides respectively 4 and 6; find their projections upon a diagonal.

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* In space, as well as in a plane.

4. A given line AB makes an angle of $30°$ with the line MN, and BC is perpendicular to AB and of length 15; find the projection of BC upon MN.

Solve this problem if the given angle be α instead of $30°$.

5. Two lines in space, of length a and b respectively, make an angle ω with each other; find the projection of b upon a line that is perpendicular to a.

6. Project the perimeter of a square upon one of its diagonals.

The inclination of a line is the ∠ which that part of the line above the "x" axis makes with positive direction of the X axis. Thus the inclination of a line may be between $0°$ & $180°$

The slope of a line is the tangent of the angle which the line makes with the positive direction of the "x" axis.

CHAPTER II

GEOMETRIC CONCEPTIONS. THE POINT

I. COÖRDINATE SYSTEMS

14. Coördinates of a point. Position, like magnitude, is relative, and can be given for a geometric figure only by reference to some fixed geometric figures (planes, lines, or points) which are regarded as known, just as magnitude can be given only by reference to some standard magnitudes which are taken as units of measurement. The position of the city of New York, for example, when given by its latitude and longitude, is referred to the equator and the meridian of Greenwich, — the position of these two lines being known, that of New York is also known. So also the position of Baltimore on the railroad from Washington to New York may be given by its distance from Washington; while a particular point in a room may be located by its distances from the floor and two adjacent walls.

If, as in the last illustration, a point is to be fixed in *space*, then *three* magnitudes must be known, referring to *three* fixed positions. If, on the other hand, the point is on a known surface, as New York or Baltimore on the surface of the earth, then only *two* magnitudes need be known, referring to two fixed positions on that surface; while if the point is on a known line, only *one* magnitude, referring to one fixed position on that line, is needed to fix its position.

These various magnitudes which serve to fix the position of a point, — in space, on a surface, or on a line, — are called the **coördinates** of the point.

In Part I of this book it will be understood that the work is restricted to a given plane surface.

15. Positive and negative coördinates. If a point lies in a given directed straight line, its position with reference to a fixed point of that line is completely determined by *one* coördinate. *E.g.*, let $X'OX$ be a given directed straight line, and let distances from O toward X be regarded as positive, then distances from O toward X' are negative. A point P in this line and 3 units from O toward X may be designated by $^+3$, where the sign $+$ gives the *direction* of the point, and the number 3 its distance, from O. Then the point P' lying 3 units on the other side of O would be designated by $^-3$.

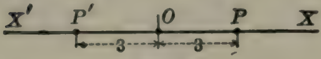

FIG. 8.

In the same way there corresponds to every real number, positive or negative, a definite point of this directed straight line; the numbers are called the **coördinates** of the points; and the point O, from which the distances are measured, is called the **origin** of coördinates.

16. Cartesian coördinates of points in a plane. Suppose two directed straight lines $X'OX$ and $Y'OY$ are given, fixed in the plane and intersecting in the point O. These two given lines are called the **coördinate axes**, $X'OX$ being the x-axis, and $Y'OY$ being the y-axis; their point of intersection O is the **origin** of coördinates. Any other two lines, parallel respectively to these fixed

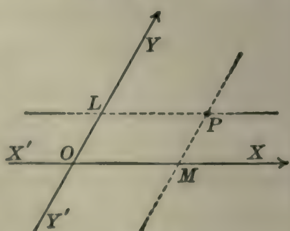

FIG. 9.

lines, and at known distances from them, will intersect in one and but one point P, whose position is thus definitely fixed. If these lines through P meet the axes in M and L respectively, then the directed distances LP and MP, *measured parallel respectively to the axes,* are the Cartesian coördinates of the point P. The distance LP, or its equal OM, is the **abscissa** of P, and is usually represented by x, while MP, or its equal OL, is the **ordinate** of P, and is usually represented by y. The point P is designated by the symbol (x, y), — often written $P \equiv (x, y)$, — the abscissa always being written first. Thus the point (4, 5) is the point for which $OM = 4$ and $MP = 5$; while the point $(^-3, 2)$ has $OM = {}^-3$ and $MP = 2$.

17. Rectangular coördinates. The simplest and most common form of Cartesian coördinate axes is that in which the angle XOY is a positive right angle; the abscissa (x) of a point is, in this case, its *perpendicular* distance from the y-axis, and its ordinate (y) is its perpendicular distance from the x-axis. This is known as the **rectangular system of coördinates.** The axes divide the entire plane into four parts called **quadrants,** which are usually designated as first (I), second (II), third (III), and fourth (IV), in the order of rotation from the positive end of the x-axis toward the positive end of the y-axis, as indicated in the accompanying figure.

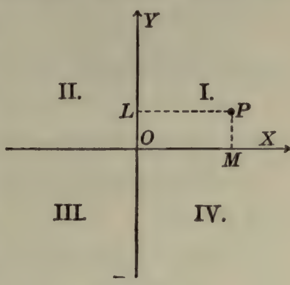

FIG. 10.

These quadrants are distinguished by the *signs* of the coördinates of the points lying within them, determined as in Art. 15, thus:*

* Compare with table in Art. 11, signs of cosine and sine.

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
abscissa x	+	−	−	+
ordinate y	+	+	−	−

Four points having numerically the same coördinates, but lying one in each quadrant, are symmetrical in pairs with regard to the origin, even though the axes are not at right angles; if, however, the axes are rectangular, then these points are symmetrical in pairs, not merely with regard to the origin as before, but also with regard to the axes, and they are severally equidistant from the origin. Because of this greater symmetry rectangular coördinates have many advantages over an oblique system.

In the following pages rectangular coördinates will always be understood unless the contrary is expressly stated.

EXERCISES

1. Plot accurately the points: $(^-1, ^-7)$*, $(^+4, ^+5)$, $(0, ^-3)$, and $(^+3, 0)$.

2. Plot accurately, as vertices of a triangle, the points: $(^-1, ^-3)$, $(^-2, ^-7)$, and $(^+4, ^+4)$. Find by measurement the lengths of the sides, and the coördinates of the middle point of each side.

3. Construct the two lines passing through the points $(a, ^-b)$ and $(^-a, b)$, and (a, b) and $(^-a, ^-b)$, respectively. What is their point of intersection? Find the coördinates of the middle point of each line.

* These minus signs are written high merely to indicate that they are signs of *quality* and not of *operation*.

4. If the ordinate of a point is 0, where is the point? if its abscissa is 0? if its abscissa is equal to its ordinate? if its abscissa and ordinate are numerically equal but of opposite signs?

5. Express each of the conditions of Ex. 4 by means of an equation.

6. The base of an equilateral triangle, whose side is 5 inches, coincides with the x-axis; its middle point is at the origin; what are the coördinates of the vertices? If the axes are chosen so as to coincide with two sides of this triangle, respectively, what are the coördinates of the vertices?

7. A square whose side is 10 inches has its diagonals lying upon the coördinate axes; find the coördinates of its vertices. If a diagonal and an adjacent side are chosen as axes, what are the coördinates of the vertices? of the middle points of the sides? of the center?

8. Find, by similar triangles, the coördinates of the point which bisects the line joining the points $(^-2, ^-7)$ and $(5, 5)$.

9. Show that the distance from the origin to the point (a, b) is $\sqrt{a^2 + b^2}$. How far from the origin is the point $(a, ^-b)$? $(^-a, b)$? $(^-a, ^-b)$?

10. Prove, by similar triangles, that the points $(4, 0)$, $(0, 3)$, and $(^-4, 6)$ lie on the same straight line.

11. Solve exercises 1 to 4, and 10, if the coördinate axes make an angle of 30°. Also if this angle is 45°.

18. Notation. In the following pages, in accordance with Art. 5, a variable point will be designated by P, and its coördinates by (x, y); so that $P \equiv (x, y)$. Several variable points under consideration at the same time will be called $P' \equiv (x', y')$, $P'' \equiv (x'', y'')$, $P''' \equiv (x''', y''')$; etc. Fixed points similarly will be designated as $P_1 \equiv (x_1, y_1)$, $P_2 = (x_2, y_2)$, etc.

II. ELEMENTARY APPLICATIONS

19. Distance between two points. *Let* $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be two given points with rectangular axes OX and OY. *To find* distance $P_1P_2 = d$ in terms of x_1, x_2, y_1, y_2.

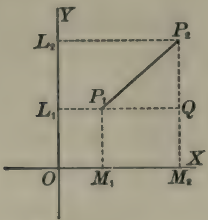

FIG. 11 *a*.

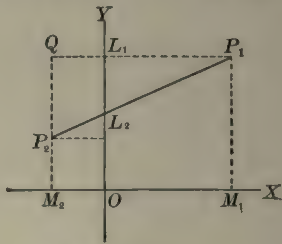

FIG. 11 *b*.

Construction: Extend the abscissa L_1P_1 of point P_1, to meet the ordinate M_2P_2 of point P_2 in Q: then in $\triangle P_1QP_2$,

N.B. $$\overline{P_1Q}^2 + \overline{QP_2}^2 = \overline{P_1P_2}^2;$$

but $$P_1Q = x_2 - x_1 \text{ and } QP_2 = y_2 - y_1;$$

hence $$\mathbf{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.} \qquad [1]^*$$

Since either of the two points may be named P_1, this formula may be expressed in words thus: *with rectangular coördinates, the square of the distance between two points is equal to the square of the difference between their abscissas plus the square of the difference between their ordinates.*

20 Direction from one point to a second. The direction of a line is indicated by the angle that it makes with the x-axis,

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* The demonstration applies to each figure. By making and examining other possible constructions the student should assure himself that the formula is entirely general. The more important formulas are printed in bold-faced type. They should be committed to memory by the learner

measured as in Art. 10. This angle is called the **inclination** of the line; it is usually found by means of its tangent, which is called the **slope*** of the line.

From this definition it follows that with rectangular axes the slope of the line from the point $P_1 \equiv (x_1, y_1)$ to the point $P_2 \equiv (x_2, y_2)$, and with an inclination ϕ, is

N. B. $$m = \tan \phi = \frac{QP_2}{P_1Q};$$

that is, $$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad [2]$$

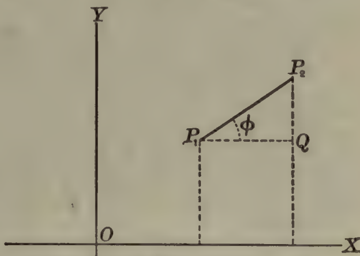

FIG. 12.

21. Parallel lines. Perpendicular lines. If two straight lines are parallel, evidently their inclinations and therefore their slopes, are equal. If the lines are perpendicular to each other, with inclinations, θ and θ' respectively, so that $\theta' = \theta + 90°$, then $\tan \theta' = -\cot \theta = -\dfrac{1}{\tan \theta}$; *i.e.*, each slope is the negative reciprocal of the other. In brief, if m_1 and m_2 are the slopes of two lines, then

N. B.

if the lines are parallel, $\quad m_2 = m_1 \quad$ [3]

if the lines are perpendicular, $\quad m_2 = \dfrac{-1}{m_1}. \quad$ [4]

EXERCISES

1. Find the distances between the points (2, 6), (4, 14), and ($^-8$, $^-8$), taken in pairs.

2. Find the distances for the points of Ex. 1, if the axes are oblique with $\omega = 30°$; if $\omega = 45°$.

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* The slope of a roof or hill has the same meaning. Thus if the slope of a hill is $\frac{3}{100}$, it rises 3 feet vertical in 100 feet horizontal.

3. Prove that the points $(^-2, ^-1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are the vertices of a parallelogram. [Two methods.]

4. Find the distance between the points $(a+b, c+a)$ and $(c+a, b+c)$; also between (a, b) and $(^-a, ^-b)$.

5. Show that the straight lines from point $(^-2, ^-4)$ to point $(3, 3)$ and from point $(1, ^-2)$ to point $(6, 5)$ are parallel.

6. A triangle has its vertices at the points $(5, 1)$, $(7, 8)$, and $(0, 10)$. Is it right-angled?

7. One end of a line whose length is 13 is at the point $(^+4, ^-8)$, the ordinate of the other end is $^-3$; what is its abscissa?

8. Express by an equation the fact that the point $P \equiv (x, y)$ is at the distance 5 from the point $(^-4, 6)$; from the point $(0, 0)$.

9. Express by an equation the fact that the point $P \equiv (x, y)$ is equidistant from the points $(^-4, 6)$ and $(8, 9)$.

10. Find the slopes of the lines which join the following pairs of points: $(3, 8)$ and $(^-1, 4)$; $(2, ^-3)$ and $(7, 9)$; $(1, ^-4)$ and $(^-3, 5)$; $(4, ^-2)$ and $(^-2, ^-1)$.

11. Show that if the axes meet at an angle ω, the formula for the distance P_1P_2 is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega},$$

and obtain formula [1] as a special case.

12. Express by an equation the fact that a straight line with slope $\frac{2}{3}$ passes through the point $(1, 4)$.

13. In eq. [2] if m is negative, what is the direction of the line? if m is positive?

14. If $m_2 = -m_1$, what is the relation of the two lines? (Cf. eq. [3]).

15. Show analytically that the line joining the points (6, 0) and (0, 4) is parallel to the line joining the points (3, 0) and (0, 2), and twice as long as the latter.

16. Show analytically that the line joining the midpoints of two sides of any triangle is parallel to, and equal to half of, the third side.

22. The area of a triangle. In accordance with Art. 17, assume *rectangular coördinates.*

Rectangular coördinates. Given a triangle with the vertices $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, and $P_3 \equiv (x_3, y_3)$; to find its area in terms of x_1, x_2, x_3, y_1, y_2, and y_3. Draw the ordinates M_1P_1,

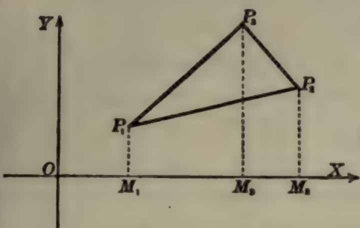

FIG. 13 *a*.

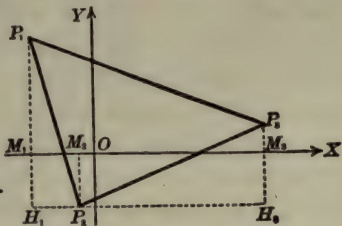

FIG. 13 *b*.

M_2P_2, and M_3P_3, — in the second figure extend M_1P_1 and M_3P_3 to meet a line through P_2 parallel to the x-axis. If Δ represents the area of the triangle, then,

$$\Delta = P_1M_1M_3P_3 + P_3M_3M_2P_2 + P_2M_2M_1P_1;$$

but $P_1M_1M_3P_3 = \frac{1}{2}(M_1P_1 + M_3P_3) \cdot M_1M_3 = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$,

and $P_3M_3M_2P_2 = \frac{1}{2}(M_3P_3 + M_2P_2) \cdot M_3M_2 = \frac{1}{2}(y_3 + y_2)(x_2 - x_3)$,

and $P_2M_2M_1P_1 = \frac{1}{2}(M_2P_2 + M_1P_1) \cdot M_2M_1 = \frac{1}{2}(y_2 + y_1)(x_1 - x_2)$.

$$\therefore \Delta = \tfrac{1}{2}\{(y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3) + (y_3 + y_1)(x_3 - x_1)\}.$$

By expanding and rearranging the second member, this latter equation may also be written in the form

V. 6

$$\Delta = \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}. \qquad [5]$$

The symmetry * in formula [5] should be carefully noted. It may be seen from the figure that with the vertices as there named, with the triangle upon the *left* as it is traced from P_1 to P_2 to P_3, the value of Δ is positive; while if the vertices be named in the reverse order, the sign of each term of eq. [5] will be reversed, and the area Δ will be negative.

23. One great advantage of the analytic method of solving problems lies in the fact that the analytic results which are obtained from the simplest arrangement of the geometric figure with reference to the coördinate axes are, from the very nature of the method, equally true for all other arrangements. Thus formulas [1], [2], and [5] can be most readily obtained if the points are all taken in quadrant I, *i.e.*, with their coördinates all positive; but because of the convention adopted concerning the signs as essential parts of the coördinates, these formulas remain true for all possible positions of P_1 and P_2. By drawing the figures and making the proofs when P_1 and P_2 are taken in various other positions, the student should assure himself of the generality of formulas [1], [2], and [5] in articles 19, 20, and 22, and of *the underlying importance in applying them of a careful interpretation of the negative number.*

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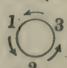
* This kind of symmetry is known as *cyclic* (or circular) symmetry. If the numbers 1, 2, and 3 are arranged thus 

, then the subscripts in the first term begin with 1 and follow the arrowheads around the circle (*i.e.*, their order is 1, 2, 3), those of the second term begin with 2 and follow the arrowheads (their order is 2, 3, 1), and those of the third term begin with 3 and follow the arrowheads.

EXERCISES

1. Find the areas of the following triangles:

(1) vertices at the points (3, 5), (4, 2), and (1, 3);

(2) vertices at the points (7, 3), ($^-4$, 6), and (3, $^-2$);

Solve without using the formula, and then verify by substituting in the formula.

2. Prove that the area of the triangle whose vertices are at the points (1, 9), (11, $^-1$), and (5, 5) is zero, and hence that these points all lie on the same straight line.

3. Are the points (2, 3), (1, $^-3$), and (3, 9) collinear?

4. Prove that the area of the triangle formed by the points (0, 0), (a, b), and (mc, md) is m times the triangle formed by the points (0, 0), (a, b), and (c, d).

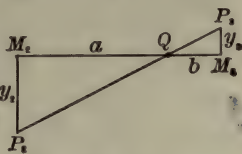


FIG. 14.

5. In Fig. 13 *b*, show that area $P_3M_3M_2P_2$ is partly positive, partly negative, so that the derivation of Art. 22 still holds.

24. **To find the coördinates of the point which divides in a given ratio the straight line from one given point to another.** Let

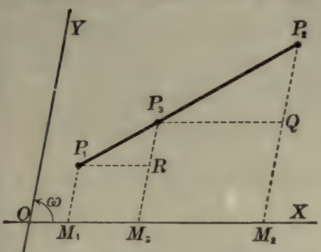


FIG. 15 *a*.

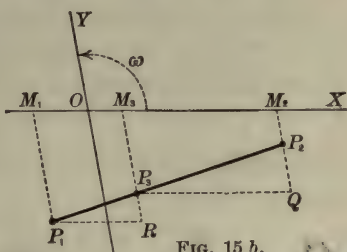


FIG. 15 *b*.

$P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be the two given points, $P_3 \equiv (x_3, y_3)$ the required point, and let the ratio of the parts into which P_3 divides P_1P_2 be $m_1 : m_2$; *i.e.*, let $P_1P_3 : P_3P_2 = m_1 : m_2$. Draw the ordinates M_1P_1, M_2P_2, M_3P_3, and through P_1 and P_3 draw lines

parallel to OX, meeting M_3P_3 and M_2P_2 in R and Q respectively.

To find $OM_3 = x_3$ and $M_3P_3 = y_3$ in terms of x_1, x_2, y_1, y_2, m_1, and m_2.

The triangles P_1RP_3 and P_3QP_2 are similar;

therefore $$\frac{P_1R}{P_3Q} = \frac{RP_3}{QP_2} = \frac{P_1P_3}{P_3P_2}.$$

But $$\frac{P_1P_3}{P_3P_2} = \frac{m_1}{m_2},$$

and $$P_1R = x_3 - x_1, \quad P_3Q = x_2 - x_3,$$

$$RP_3 = y_3 - y_1, \quad QP_2 = y_2 - y_3.$$

[In Fig. 15 *b*, x_1, y_1, y_2, and y_3 are negative.]

Therefore $$\frac{x_3 - x_1}{x_2 - x_3} = \frac{y_3 - y_1}{y_2 - y_3} = \frac{m_1}{m_2};$$

P.D whence $$x_3 = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \text{ and } y_3 = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}. \qquad [6]$$

The above reasoning applies equally well whatever the value of ω (the angle made by the coördinate axes), hence formulas [6] hold whether the axes be rectangular or oblique.

If the dividing point P_3 is not between P_1 and P_2, but upon P_1P_2 extended, then segments P_1P_3 and P_3P_2 have opposite signs, and the ratio $m_1 : m_2$ is negative. In such a case point P_3 is said to divide P_1P_2 *externally* in the ratio $m_1 : m_2$.

Corollary. If P_3 is the middle point of P_1P_2, then $m_1 = m_2$, and formulas [6] become

M. C $$x_3 = \frac{x_1 + x_2}{2}, \qquad y_3 = \frac{y_1 + y_2}{2}; \qquad [7]$$

i.e., *the abscissa of the middle point of the line joining two given points is half the sum of the abscissas of those points, and the ordinate is half the sum of their ordinates.*

EXERCISES

1. By means of an appropriate figure, derive formulas [7] independently of [6].

2. The point $P_3 \equiv (2, 3)$ is one third of the distance from the point $P_1 \equiv (^-1, 4)$ to the point $P_2 \equiv (x_2, y_2)$; to find the coördinates of P_2.

Solution. Here P_1 and P_3 are given, with $x_1 = -1$, $y_1 = 4$, $x_3 = 2$, $y_3 = 3$, also $m_1 = 1$, and $m_2 = 2$; therefore, from [6],

$$2 = \frac{x_2 + 2(-1)}{1+2}, \text{ and } 3 = \frac{y_2 + 2(4)}{1+2},$$

which give $x_2 = 8$ and $y_2 = 1$; therefore the required point P_2 is (8, 1).

3. Find the points of trisection of the line joining (2, 3) to (4, $^-5$).

4. Find the point which divides the line from (1, 3) to ($^-2$, 4) externally into segments whose numerical ratio is 3 : 4.

Solution. Here $x_1 = 1$, $y_1 = 3$, $x_2 = -2$, $y_2 = 4$, $m_1 = 3$, and $m_2 = 4$, but the point of division being an external one, the two segments are oppositely directed; therefore one of the numbers 3 or 4, say 4, must have the minus sign prefixed to it. Substituting these values in [6],

$$x_3 = \frac{3(-2) - 4(1)}{3-4} = 10, \text{ and } y_3 = \frac{3(4) - 4(3)}{3-4} = 0;$$

the required point is, therefore, $P_3 \equiv (10, 0)$.

5. Solve Ex. 4 directly from a figure, without using [6].

6. Find the points which divide the line from (2, 10) to (4, 14) internally and externally into segments that are in the ratio 2 : 3.

7. A line AB is produced to C, so that $BC = \frac{1}{2} AB$; if the points A and B have the coördinates (5, 6) and (7, 2), respectively, what are the coördinates of C?

8. Prove, by means of Art. 24, that the median lines of a triangle meet in a point, which is for each median the point of trisection nearest the side of the triangle.

25. **Analytic Geometry** is the study of geometric figures by algebraic methods. It may conveniently be divided into Plane Analytic Geometry, which concerns only figures in a given plane surface, and is treated in Part I of this book; and Solid Analytic Geometry, which concerns space figures, and is taken up in Part II.

The fundamental basis of analytic geometry is the representation of a geometric point by means of algebraic numbers, which has been the subject of the present chapter. The next important step is to find the relation that exists between an algebraic equation and a geometric curve, or locus. This will be taken up in the next chapter.

EXAMPLES ON CHAPTER II

1. Find the area of the quadrilateral whose vertices are the points (0, 0), (5, 0), (2, 3), and (4, 5). Draw the figure.

2. Find the lengths of the sides and the altitude of the isosceles triangle (0, 5), (5, 0), (9, 9). Find the area by two different methods, so that the results will each be a check on the other.

3. Find the coördinates of the point that divides the line from (2, 3) to ($^-1$, $^-6$) in the ratio 6 : 8; in the ratio 2 : $^-3$; in the ratio 3 : $^-2$. Draw each figure.

4. One extremity of a straight line is at the point ($^-3$, 4), and the line is divided by the point (1, 6) in the ratio 4 : 6; find the other extremity of the line.

5. The line from $(^-6, ^-2)$ to $(3, ^-1)$ is divided in the ratio $5:4$; find the distance of the point of division from the point $(^-4, 6)$.

6. Show analytically that the figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram.

7. Show that the points $(3, ^-1)$, $(\sqrt{3}, \sqrt{7})$, and $(\sqrt{6}, 2)$ are equidistant from the origin.

8. Show that the points $(1, 1)$, $(^-1, ^-1)$, and $(a, ^-a)$ form an isosceles triangle. Find the slopes of its sides.

9. Prove analytically that the diagonals of a rectangle are equal.

10. Show that the points $(0, ^-a)$, $(2a, a)$, $(0, 3a)$, and $(^-2a, a)$ are the vertices of a square.

11. Express by an equation that the point (h, k) is equidistant from $(^-1, 1)$ and $(1, 2)$; from $(1, 2)$ and $(1, ^-2)$. Then show that the point $(\frac{3}{4}, 0)$ is equidistant from $(^-1, 1)$, $(1, 2)$, and $(1, ^-2)$.

12. Prove analytically that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices

13. Three vertices of a parallelogram are $(1, 2)$, $(^-5, ^-3)$, and $(7, ^-6)$; what is the fourth vertex?

14. The center of gravity of a triangle is at the point in which the medians intersect. Find the center of gravity of the triangle whose vertices are $(2, 3)$, $(4, ^-5)$, and $(3, ^-6)$. [Cf. Ex. 8, p. 32].

15. The line from (x_1, y_1) to (x_2, y_2) is divided into four equal parts; find the points of division.

16. Prove analytically that the two straight lines which join the middle points of the opposite sides of a quadrilateral mutually bisect each other.

17. Prove that (2, 10) is on the line joining the points (0, 4) and (4, 16), and is equidistant from them.

18. If the angle between the axes is 60°, find the perimeter of the triangle whose vertices are (2, 2), $({}^-7, {}^-1)$, and $({}^-1, 5)$. Plot the figure.

19. Show analytically that the line joining the middle points of two sides of a triangle is half the length of the third side.

20. A point is 7 units distant from the origin and is equidistant from the points (4, 2) and $({}^-4, {}^-2)$; find its coördinates.

21. Prove that the points $(a, b+c)$, $(b, c+a)$, and $(c, a+b)$ lie on the same straight line. [Cf. Ex. 2, p. 29].

22. What is the ordinate of a point whose abscissa is c, which lies on the line from the origin to the point (a, b)?

23. Find five pairs of values of x and y that satisfy each of the following equations, and plot the corresponding points:

(1) $3x + 2y + 6 = 0$. (3) $2x + 7 = 3y$.

(2) $x^2 + y^2 = 9$. (4) $xy = 4$.

The locus of an equation containing two variables is the line or curve or set of lines which contains all the pts. whose coordinates satisfy the equation, and contains no other points

The locus of an equation is the path traced by a pt. whose co-ordinates satisfy the equation.

CHAPTER III

THE LOCUS AND THE EQUATION

26. The locus of an equation. A pair of numbers (x, y) is represented geometrically by a point in a plane. If these two numbers (x, y) are variables, but connected by an equation, then this equation can, in general, be satisfied by an infinite number of pairs of values of x and y, and each pair may be represented by a point. These points will not, however, be scattered indiscriminately over the plane, but will all lie in a definite curve, whose form depends only upon the nature of the equation under consideration; and this curve will contain no points except those whose coördinates are pairs of values which, when substituted for x and y, satisfy the given equation. This curve is called the **locus** or **graph** of the equation; and the first fundamental problem of analytic geometry is to find, for a given equation, its graph or locus.

27. Illustrative examples: rectangular coördinates. (1) *Given the equation $x+5=0$, to find its locus.* This equation is satisfied by the value -5 for x, whatever the value given to y. Now such points as

$$P_1 \equiv (-5, 2), \quad P_2 \equiv (-5, 3), \quad P_3 \equiv (-5, -3), \text{ etc.,}$$

all lie on the line MN, parallel to the y-axis, and at the distance 5 on the negative side of it,—this line extending indefinitely in both directions. Moreover, each point of MN has for

its abscissa -5, hence the coördinates of each of its points satisfy the equation $x+5=0$. In the chosen system of coördinates, the line MN is the locus of the equation $x+5=0$.

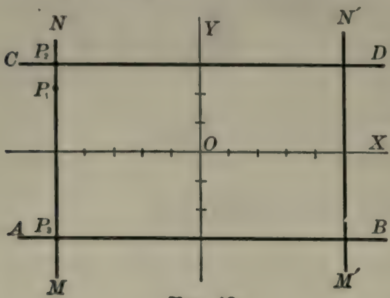


FIG. 16

Similarly, the equation $x-5=0$ is satisfied by any pair of values of which x is 5, such as (5, 2), (5, 3), (5, 4), etc.; all the corresponding points lie on a straight line $M'N'$, parallel to the y-axis, at the distance 5 from it, and on its positive side; *i.e.*, $M'N'$ is the locus of the equation

$$x-5=0.$$

(2) *Given the equations* $y \pm 3 = 0$, *to find their loci.* By the same reasoning as in (1) it may be shown that the locus of the equation $y+3=0$ is the straight line AB, parallel to the x-axis, situated at the distance 3 from it, and on its negative side. Also that the locus of the equation $y-3=0$ is CD, a line parallel to the x-axis, at the distance 3 from it, and on its positive side.

More generally it is evident that *in Cartesian coördinates (rectangular or oblique), an equation of the first degree, and containing but one variable, represents a straight line parallel to one of the coördinate axes.*

(3) *Given the equation* $3x-2y+12=0$, *to find its locus.* In this equation *both* the variables appear. By assigning any definite value to either one of the variables, and solving the equation for the other, a pair of values that will satisfy the equation is obtained. Thus the following pairs of values are found:

$x_1 = 0,\ y_1 = 6$ $\qquad x_5 = -1,\ y_5 = 4\frac{1}{2}$

$x_2 = 1,\ y_2 = 7\frac{1}{2}$ $\qquad x_6 = -2,\ y_6 = 3$

$x_3 = 2,\ y_3 = 9$ $\qquad x_7 = -3,\ y_7 = 1\frac{1}{2}$

$x_4 = 3,\ y_4 = 10\frac{1}{2}$ $\qquad x_8 = -4,\ y_8 = 0$

$\cdots\cdots \qquad \cdots\cdots$

$x = +\infty,\ y = +\infty$ $\qquad x = -\infty,\ y = -\infty$

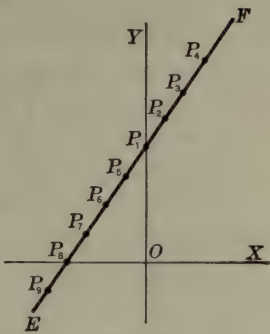


Fig. 17.

Plotting the corresponding points

$$P_1,\ P_2,\ P_3,\ P_4 \cdots,$$

where $P_1 \equiv (x_1, y_1) \equiv (0, 6)$,

$P_2 \equiv (x_2, y_2) \equiv (1, 7\frac{1}{2})$, etc.,

they are all found to lie on the straight line EF, which is the locus of the equation $3x - 2y + 12 = 0$.

In Chap. V, it will be shown that, in Cartesian coördinates, an equation of the first degree in two variables always represents a straight line

EXERCISES

Plot the loci of the following equations:

1. $x = 0$.

2. $y = 0$.

3. $mx = 0$.

4. $7x = 3$.

5. $2y - 5 = 0$.

6. $x + y = 0$.

7. $3x + 2 = y$.

8. $2x^2 + 2y^2 = 4$.

9. $x + y = 8$.

10. $x - y = 0$.

11. $x^2 - y^2 = 16$.

12. $x^2 + 2y^2 = 4$.

13. $v = 32t$.

14. $x = 3y + 2$.

15. $t^2 + s^2 = 25$.

16. $u^2 - v = 0$.

17. $s = 16t^2$.

18. $\frac{x}{3} + \frac{y}{2} = 1$.

19. $xy = 4$.

20. $x^2 = 4y$.

21. $y = -x^3$.

28. The locus of an equation. By the process illustrated above, of constructing a curve from its equation, the first conception of a locus is obtained, **viz.**:

(1) *The locus of an equation containing two variables is the line, or set of lines, which contains all the points whose coördinates satisfy the given equation, and which contains no other points.* It is the *place where* all the points, and only those points, are found whose coördinates satisfy the given equation.

A second conception of the locus of an equation comes directly from this one, for the line or set of lines may be regarded as the *path traced* by a point which moves along it. The path of the moving point is determined by the condition that its coördinates for every position through which it passes must satisfy the given equation. Thus the line EF [the locus of eq. (3), Art. 27] may be regarded as the path traced by the point P, which moves so that its coördinates (x, y) always satisfy the equation $$3x - 2y + 12 = 0.$$

Thus arises a second conception of a locus, viz.:

(2) *The locus of an equation is the path traced by a point which moves so that its coördinates always satisfy the given equation.*

In either conception of a locus, the essential condition that a point shall lie on the locus of a given equation is, that *the coördinates of the point when substituted respectively for the variables of the equation, shall satisfy the equation*; and in order that a curve may be the locus of an equation, it is necessary that *there be no other points than those of this curve whose coördinates satisfy the equation.*

29. Classification of loci. The form of a locus depends upon the nature of its equation; the curve may therefore be classified according to its equation, an algebraic curve being one whose equation is algebraic, and a transcendental curve one whose equation is transcendental. In particular, the **degree** *of an algebraic curve* is defined to be the same as the degree of its equation. The following pages are concerned chiefly with algebraic curves of the first and second degrees.

30. Construction of loci. Discussion of equations. The process of constructing a locus by plotting separate points, and then connecting them by a smooth curve, is only approximate, and is long and tedious. It may often be shortened by a consideration of the peculiarities of the given equation, such as symmetry, the limiting values of the variables for which both are real, etc. Such considerations will often show the general form and limitations of the curve; and, taken together, they constitute a *discussion of the equation.*

The points where a locus crosses the coördinate axes are almost always useful; they are given by their distances from the origin along the respective axes, called the **intercepts** of the curve.

The following examples may serve to illustrate these conceptions.

(1) *Discussion of the equation* $3x - 2y + 12 = 0.$

Intercepts: if $x = 0$, then $y = 6$; hence the y-intercept is 6 (see Fig. 17); if $y=0$, then $x = -4$; hence the x-intercept is -4.

The equation may be written:

$$x = \tfrac{2}{3}y - 4,$$

which shows that as y increases continuously from 0 to ∞, x increases continuously from -4 to ∞; therefore the locus passes from the point P_8 through the point P_1, and then recedes indefinitely from both axes in the first quadrant.

Written as above, the equation also shows that as y decreases from 0 to $-\infty$, x also decreases from -4 to $-\infty$; therefore the locus passes from P_8 into the third quadrant, receding again indefinitely from both axes. Since for every value of y, x takes but one value (*i.e.*, each value of y corresponds to but one point on the curve), therefore the locus consists of a single branch. The *proof* that the locus of any first-degree equation, in two variables, is a straight line is given in Chap. IV.

(2) *Discussion of the equation* $x^2+y^2=a^2$.

Intercepts: if $x=0$, then $y=\pm a$, and if $y=0$, then $x=\pm a$; hence for each axis there are *two* intercepts, each of length a, and on opposite sides of the origin; *i.e.*, four positions of the tracing point are: $A\equiv(a, 0)$, $A'\equiv(^-a, 0)$, $B\equiv(0, a)$, and $B'\equiv(0, ^-a)$.

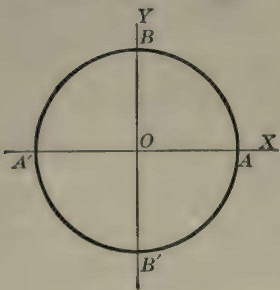


FIG. 18.

This equation may also be written

$$y=\pm\sqrt{a^2-x^2},$$

which shows that every value of x gives two corresponding values of y which are numerically equal, but of opposite sign; the locus is, therefore, symmetrical with regard to the x-axis.

It shows also that, corresponding to any value of x numerically greater than a, y is imaginary; the tracing point, therefore, does not move farther from the y-axis than $\pm a$, *i.e.*, farther than the points A and A'. Moreover, as x increases from 0 to a, y remains real and changes gradually from $+a$ to 0, or from $-a$ to 0; *i.e.*, the tracing point moves continuously from B to A, or from B' to A.

Again, if x decreases from 0 to $-a$, y remains real and changes continuously from $+a$ to 0, or from $-a$ to 0; *i.e.*, the tracing point moves continuously from B to A' or from B' to A'.

Similarly, the equation may be written $x=\pm\sqrt{a^2-y^2}$, which shows that the curve is symmetrical with regard to the y-axis, and that the tracing point does not move farther than $\pm a$ from the x-axis.

From these facts it follows that this locus is a closed curve of only one branch. It is a circle of radius a, with its center at the origin; this curve will be studied in detail in Chap. V

(3) *Given the equation $y^2 = 4x$, to find its locus.* This equation may be solved by the forms

$$y = \pm\sqrt{4x}, \quad \text{and} \quad x = \frac{y^2}{4},$$

from which it is clear that for each value given to x there are two values of y, numerically equal but of opposite sign; hence the locus is symmetrical with respect to the x-axis; while for each value given to y corresponds only one value of x. Again, for negative values of x there are no corresponding real values for y. Hence the locus lies wholly on the positive side of the y-axis. The intercepts are both zero. The following points are points of the locus:

$P_1 \equiv (0, 0)$, $P_2 \equiv (1, 2)$,

$P_3 \equiv (1, -2)$, $P_4 = (2, 2.8\cdots)$,

$P_5 \equiv (2, -2.8\cdots)$, $P_6 \equiv (4, 4)$,

and $P_n \equiv (+\infty, \pm\infty)$.

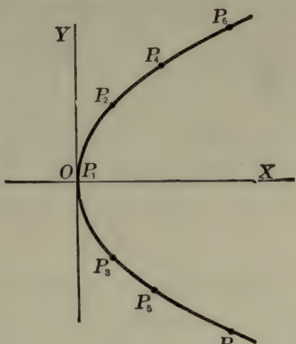


FIG 19.

All these points are found to lie on the curve as plotted in Fig. 19. This curve is called a *parabola*, and will be studied in a later chapter.

The parabola is one of the curves obtained by the intersection of a circular cone and a plane. It will be shown in Chap. VIII that in Cartesian coördinates the locus of any algebraic equation in two variables and of the second degree is a "conic section."

(4) *Discussion of the equation $y^2 = (x-2)(x-3)(x-4)$.*

Intercepts: if $x = 0$, then y is imaginary; if $y = 0$, then $x = 2, 3$, or 4; hence the locus crosses the x-axis at the three points $A \equiv (2, 0)$, $B \equiv (3, 0)$, and $C \equiv (4, 0)$, and it does not

cut the y-axis at all. Moreover, since y is imaginary if x is negative, the locus lies wholly on the positive side of the y-axis.

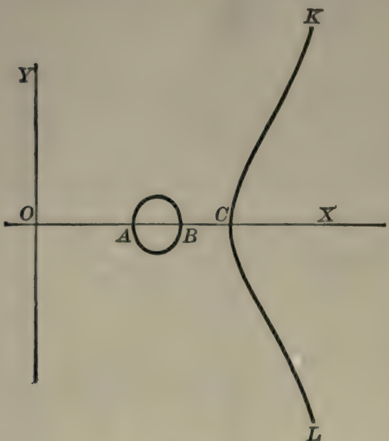


Fig. 20.

This locus is symmetrical with regard to the x-axis; it has no point nearer to the y-axis than A; between A and B it consists of a closed branch; and it has no real points between B and C, but is again real beyond C. The entire locus consists, then, of a closed oval, and of an open branch which recedes indefinitely from both axes. (See Fig. 20.)

In discussing an equation it is well to consider

(1) intercepts;

(2) symmetry, with respect to each axis;

(3) limiting values, if any, of each variable;

(4) whether the locus is a closed or open curve.

EXERCISES

Construct and discuss the loci of the following equations:

1. $\frac{x^2}{9} - \frac{y^2}{4} = 1.$

2. $x^2 - y^2 = 4a^2.$

3. $\frac{x^2}{9} + \frac{y^2}{4} = 1.$

4. $x^2 - y^2 = 0.$

5. $x^2 + y^2 = 0.$

6. $x^2 - 4y^2 = 0.$

31. The equation of a locus. The second fundamental problem of analytic geometry is the reverse of the first [cf. Art. 26], and is usually more difficult. It is to find, for a given geo-

metric figure, or locus, the corresponding equation, *i.e.*, the equation which shall be satisfied by the coördinates of every point of the given locus, and which shall not be satisfied by the coördinates of any other point. The geometric figure may be given in two ways, viz.:

(1) As a figure with certain known properties; and

(2) As the path of a point which moves under known conditions.

In the latter case the path is usually unknown, and the complete problem is, first to find the equation of the path, and then from this equation to find the properties of the curve.

The two ways by which a locus may be "given" correspond to the two conceptions of a locus mentioned in Art. 28, and they lead to somewhat different methods of obtaining the equation. The first method may be exemplified clearly by considering the familiar cases of the straight line and the circle.

32. Equation of a locus with known properties. Let $P_1 \equiv (3, 2)$ and $P_2 \equiv (12, 5)$ be two given points; and let $P \equiv (x, y)$ be *any* other point in the straight line through P_1 and P_2.

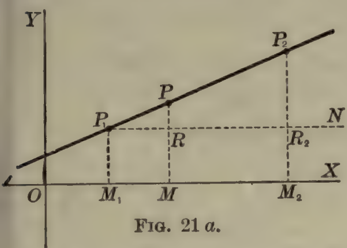


FIG. 21 *a*.

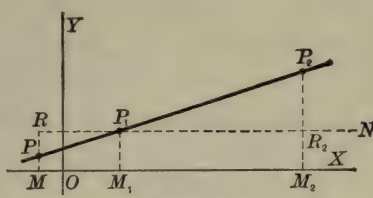


FIG. 21 *b*.

The slope of the line is the same, whatever two points are taken to determine it; hence

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}. \qquad \text{[Art. 20]}$$

On substituting the respective values for the coördinates,

$$\frac{5-2}{12-3}=\frac{y-2}{x-3},$$

which reduces to $\quad 3y-x-3=0. \quad (1)$

This is the required equation of the straight line through P_1 and P_2, because it fulfills both the requirements of the definition [cf. Art. 28 (1)]; *i.e.*, it *is* satisfied by the coördinates of *any* (*i.e.*, of every) point of this line, because x, y are the coördinates of any such point; and it *is not* satisfied by the coördinates of any point which is not on this line; for let $P_3 \equiv (x_3, y_3)$ be any point not on the line through P_1 and P_2, the ordinate M_3P_3 will meet P_1P_2 in some point $P_4 \equiv (x_4, y_4)$, for which $x_4 = x_3$ but $y_4 \neq y_3$. Since P_4 is *on* the line P_1P_2, its coördinates satisfy eq. (1), therefore

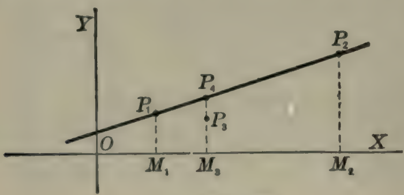


FIG. 22.

$$3y_4-x_4-3=0, \quad\text{and}\quad y_4=\frac{x_4+3}{3}.$$

$$\therefore \qquad y_3 \neq \frac{x_3+3}{3}, \text{ so that } 3 y_3-x_3-3 \neq 0;$$

i.e., the coördinates of P_3 do not satisfy the equation

$$3y-x-3=0.$$

NOTE. If the coördinates of any point *on* the straight line P_1P_2 are substituted for x and y in eq. (1) the first member will be equal to zero. Substituting the coördinates of a point on one side of the line, the first member will be negative; of a point on the other side, it will be positive. The line is the boundary which separates the part of the plane for which $3y-x-3$ is negative, the *negative side* of the line, from the part for which this function is positive, the *positive side*.

Again, let $P \equiv (x, y)$ be *any* point on a circle of radius 5 about the origin as center.

Then, by the well-known property of right triangles,

$$\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2,$$

i.e., $$x^2 + y^2 = 25;$$

and this equation is not satisfied by the coördinates of any point not on the circle. Hence this is the equation of the circle.

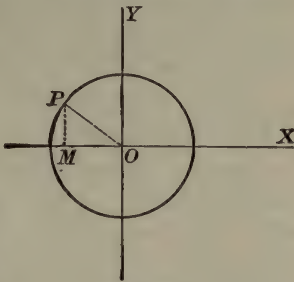


FIG. 23.

EXERCISES

1. Find the equation of the straight line through the two points $(^-1, ^-7)$ and $(11, 6)$; through the points $(2, ^-5)$ and $(8, 3)$.

2. Find the equation of the straight line through the two points $(4, 6)$ and $(^-4, ^-6)$. Through what other point does this line pass? Does the equation show this fact?

3. Find the equation of the straight line through the point $(10, ^-14)$, and making an angle of $45°$ with the x-axis; making the angle $-45°$ with the x-axis. On which side is the origin?

4. Find the equation of the line through the point $(6, 2)$, and making the angle $150°$ with the x-axis.

5. With rectangular coördinates, find the equation of the circle of radius 5, which passes through the origin, and has its center on the y-axis. Is its positive side outside or inside?

33. Equation of a locus traced by a moving point. In the exercises given above, the geometric figure in each case was completely known; and, in obtaining its equation, use was made of the known properties of straight lines, similar triangles, right triangles, and trigonometric functions. In only a few cases,

however, is the curve so completely known; in a large class of important problems, the curve is known merely as the path traced by a point which moves under given conditions or laws. Such a curve, for instance, is the path of a cannon ball, or other projectile, moving under the influence of a known initial force and the force of gravity. Another such curve is that in which iron filings arrange themselves when acted upon by known magnetic forces. The orbits of the planets and other astronomical bodies, acting under the influence of certain centers of force, are important examples of this class of "given loci."

In such problems as these, the method used in Art. 32 cannot, in general, be applied. A method that can often be employed, after the construction of an appropriate figure, is:

(1) *From the figure, express the known law, under which the point moves, by means of an equation involving geometric magnitudes;* this may be called the "geometric equation."

(2) *Find for each geometric magnitude used an equivalent algebraic value,* expressed in terms of the coördinates of the moving point and given constants.

(3) *Replace each geometric magnitude by its equivalent algebraic value,* thus obtaining the "algebraic equation."

(4) *Simplify the algebraic equation.* The result is "the" equation of the locus.

34. Equation of a circle: second method. To illustrate this second method of finding the equation of a locus, consider the circle as the path traced by a point which moves so that it is always at a given constant distance from a fixed point. From this definition, find its equation.

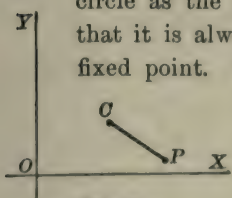


FIG. 24.

Let $C \equiv (3, 2)$ be the given fixed point, and let $P \equiv (x, y)$ be a point that moves so as to be always at the distance $2\frac{1}{2}$ from C.

Then $\qquad CP = \frac{5}{2}$, [geometric equation]

but $\qquad CP = \sqrt{(x-3)^2 + (y-2)^2}$, [algebraic value]

$\therefore \qquad \sqrt{(x-3)^2 + (y-2)^2} = \frac{5}{2}$; [algebraic equation]

i.e., $\qquad (x-3)^2 + (y-2)^2 = \frac{25}{4}$;

hence $\qquad 4x^2 + 4y^2 - 24x - 16y + 27 = 0$,

which is the required equation of the locus.

EXERCISES

1. Find the equation of the path traced by a point which moves so that it is always at the distance 5 from the point (4, 0). Trace the locus.

2. Find the equation of the path traced by a point which moves so that it is always equidistant from the points $(^-2, 3)$ and (7, 5). On which side is the origin?

3. A line is 3 units long; one end is at the point $(2, ^-3)$. Find the locus of the other end.

4. A point moves so as to be always equidistant from the y-axis and from the point (8, 0). Find the equation of its path, and then trace and discuss the locus from its equation.

5. A point moves so that the sum of its distances from the two points $(0, \sqrt{5})$, $(0, ^-\sqrt{5})$ is always equal to 8. Find the equation of the locus traced by this moving point.

6. A point moves so that the difference of its distances from the two points $(0, \sqrt{5})$, $(0, ^-\sqrt{5})$ is always equal to 2. Find the equation of the locus traced by this moving point.

35. General Theorems. A few general theorems about loci and their equations are of constant use.

The locus of an equation remains unchanged: (α) by any transposition of the terms of the equation; and (β) by multiplying both members of the equation by any finite constant.

(α) If in any equation the terms are transposed from one member to the other in any way whatever, the locus of the equation is not changed thereby; for the coördinates of all the points which satisfied the equation in its original form, and only those coördinates, satisfy it after the transpositions are made. [See Art. 28 (1)].

(β) If both members of an equation are multiplied by any finite constant k, its locus is not changed thereby. For if the terms of the equation, after the multiplication has been performed, are all transposed to the first member, that member may be written as the product of the constant k and a factor containing the variables. This product will vanish if, and only if, its second factor vanishes; but this factor will vanish if, and only if, the variables which it contains are the coördinates of points on the locus of the original equation. Hence the coördinates of all points on the locus of the original equation, and only those coördinates, satisfy the equation *after* it has been multiplied by k; hence the locus remains unchanged if its equation is multiplied by a finite constant.

E.g., (*a*) $3x^2+2y^2-2x+3=0$ and $3x^2+2y^2=2x-3$ have the same locus.

(*b*) $5x-7y+6=0$ and $15x-21y+18=0$ have the same locus.

NOTE. The theorem of this article is of great service in studying the properties of curves represented by equations. Certain standard equations are studied thoroughly, and the properties of the corresponding curves found. Other equations shown to differ from the standard equations by a constant factor only, are then also completely known.

36. Points of intersection of two loci. Since the points of intersection of two loci are points on *each* locus, therefore the coördinates of these points must satisfy each of the two equations; moreover, the coördinates of no other points can satisfy *both* equations. Hence, *to find the coördinates of the points of*

intersection of two curves, regard their equations as simultaneous and solve.

E.g., Find the coördinates of the points of intersection, P_1 and P_2, of the loci of $x - 2y = 0$, and $y^2 = x$. The point of intersection $P_1 \equiv (x_1, y_1)$ is on *both* curves,

$\therefore \quad x_1 - 2y_1 = 0,$

and $\quad y_1^2 = x_1.$*

Solving these two equations $\quad x_1 = 0$, or 4,

and $\quad y_1 = 0$, or 2;

i.e., $\quad P_1 \equiv (4, 2)$

and $\quad P_2 \equiv (0, 0)$

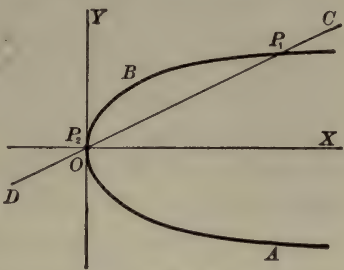


FIG. 25.

are two points, the coördinates of which satisfy each of the two given equations; therefore they are the points of intersection of the loci of these equations.

EXERCISES

Find the points of intersection of the following pairs of curves:

1. $\begin{cases} 3x - 4y + 16 = 0, \\ 3x - y - 7 = 0. \end{cases}$

2. $\begin{cases} x + y = 4, \\ x - y = 4. \end{cases}$

3. $\begin{cases} y = 3x + 2, \\ x^2 + y^2 = 4. \end{cases}$

4. $\begin{cases} 2y - 5x = 0, \\ x^2 - y^2 = 6. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 9, \\ x^2 + 6xy + y^2 = 0. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 25, \\ 2xy = 9. \end{cases}$

* If x and y are regarded as the coördinates of the point of intersection, the subscripts may be omitted here.

7. $\begin{cases} y^2 = 4x, \\ y - x = 0. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 16, \\ x^2 - 2y^2 = 1. \end{cases}$

8. $\begin{cases} x + y = 2b, \\ b^2x^2 + a^2y^2 = a^2b^2. \end{cases}$

10. $\begin{cases} x^2 = 4y, \\ y - x = 8. \end{cases}$

37. Product of two or more equations. *Given two or more equations with their second members zero;* the product of their first members, equated to zero, has for its locus the combined loci of the given equations.*

This follows at once from the fundamental relation between an equation and its locus [see Art. 28 (1)], by reasoning as illustrated in the following example.

Let $$x + y = 0 \tag{1}$$
and $$x - y = 0 \tag{2}$$
be two given equations, whose loci are respectively the straight lines CD and AB, bisecting the angles between the axes. To show that the equation

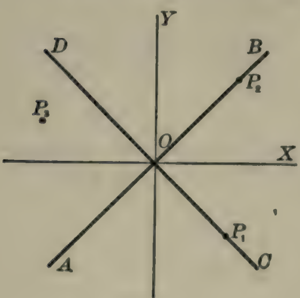


FIG. 26.

$$(x + y)(x - y) = 0$$
i.e., $$x^2 - y^2 = 0 \tag{3}$$
has for its locus both these lines.

Proof. If $P_1 \equiv (x_1, y_1)$ is any point on CD,
then $$x_1 + y_1 = 0,$$
hence $$(x_1 + y_1)(x_1 - y_1) = 0;$$
therefore P_1 is a point on the locus of equation (3).

* If equations whose second members are *not* zero are multiplied together, member by member, the resulting equation is not in general satisfied by any points of the loci of the given equations except those in which they intersect each other; the new equation therefore represents a locus through the points of intersection of the loci of the given equations.

But P_1 is *any* point on CD, hence every point on CD is a point of the locus of equation (3).

In the same way it is shown that AB belongs to the locus of equation (3).

Moreover, if $P_3 \equiv (x_3, y_3)$ be any point not on AB nor on CD, then $x_3 + y_3 \neq 0$, and $x_3 - y_3 \neq 0$, hence

$$(x_3 + y_3)(x_3 - y_3) \neq 0;$$

i.e., P_3 does not belong to the locus of equation (3).

Hence the locus of equation (3) contains the loci of equation (1) and (2), but contains no other points.

The above theorem may be stated briefly thus: if u, v, w, etc., be any functions of two variables, then the equation $uvw \cdot \cdots = 0$ has for its locus the combined loci of the equations $u = 0$, $v = 0$, $w = 0$, etc.; and conversely.

Note. When possible, factoring the first member of an equation, whose second member is zero, simplifies the work of finding the locus of the given equation.

EXERCISES

What loci are represented by the following equations?

1. $xy = 0$.
2. $\frac{x^2}{9} - \frac{y^2}{4} = 0$.
3. $3x^2 + 2xy - 7x = 0$.
4. $2xy^2 - 5x^2y = 0$.
5. $x^2 + 4x + 4 = 0$.
6. $(x^2 + y^2 - 10)(y^2 - 10x^2) = 0$.

38. Locus represented by the sum of two equations. Given the equations

$$3x - 4y - 1 = 0 \quad \text{and} \quad x + 2y - 7 = 0,$$

representing respectively the loci AB and CD.

Required to find the locus of their sum, *i.e.*, of

$$4x - 2y - 8 = 0.$$

Plotting, the locus LM is found to pass through the point of intersection P of the two given loci (Fig. 27).

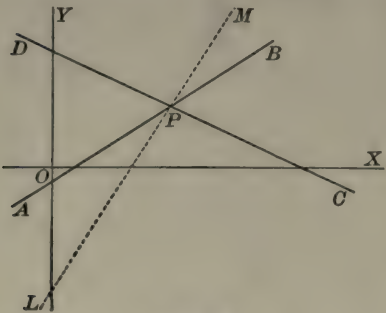


FIG. 27.

Again, given equations

$$2y - x = 0 \text{ and} \quad (1)$$

$$y^2 - x = 0 \quad (2)$$

representing respectively the loci AB and CP_1P_2D which intersect in P_1 and P_2 (Fig. 25). Then the locus of the equation

$$y^2 - x + (2y - x) = 0,$$

i.e., $y^2 + 2y - 2x = 0$, (3)

must represent some curve through P_1 and P_2.

For, let $P_1 \equiv (x_1, y_1)$ be one of the common points;

then $\quad 2y_1 - x_1 = 0 \quad$ and $\quad y_1^2 - x_1 = 0,$

since P_1 is on each curve.

$$\therefore \quad y_1^2 - x_1 + 2y_1 - x_1 = 0,$$

i.e., P_1 satisfies equation (3), and therefore the locus of (3) passes through P_1. Similar reasoning would show that the locus of equation (3) passes through every other point in which the loci of equations (1) and (2) intersect each other. In precisely the same way it may be proved generally that *the locus of the sum of two equations passes through all the points in which the loci of the two given equations intersect each other.*

If either of the given equations (1) or (2) had been multiplied by any constant factor before adding, the above reasoning would still have led to the same conclusion; in fact, this theorem may be briefly, and more generally, stated thus: *if u and v are any functions of the two variables x and y, and k is any constant, then the locus of*

$$u + kv = 0$$

passes through every point of intersection of the loci of

$$u = 0 \quad \textit{and} \quad v = 0.$$

NOTE. This theorem is of great service in finding the points of intersection of the loci of two equations, as in Art. 36.

E.g., the points of intersection of the loci of equations (1) and (2) above, are also on the locus of

$$y^2 - x - (2y - x) = 0, \text{ i.e., of } y^2 - 2y = 0, \text{ i.e., } y(y - 2) = 0,$$

hence are on the lines $y = 2$ and $y = 0$. Therefore, the points are

$$P_1 = (4, 2) \text{ and } P_2 = (0, 0).$$

Geometrically, this means that instead of fixing the points by the given parabola and oblique straight line, they are fixed by the straight lines parallel to the axes. [Cf. Art. 144].

39. Fundamental problems of analytic geometry. The elementary applications already considered have indicated how algebra may be applied to the solution of geometric problems. Points in a plane have been identified with pairs of numbers, — the coördinates of those points, — and it has been seen that definite relations between such points correspond to definite relations between their coördinates.

It has been found also that the relation between points, which consists in their lying on a definite curve, corresponds to the relation between their coördinates, which consists in their satisfying a definite equation. From this fact have arisen the two fundamental problems of analytic geometry:

I. *Given an equation, to find the corresponding geometric curve, or locus.*

II. *Given a geometric curve, to find the corresponding equation.*

From this relation between a curve and its equation a third problem arises:

III. *To find the properties of the curve from those of its equation.*

The remaining chapters of Part I will be concerned chiefly with the third problem. In this application of analytic methods, however, only algebraic equations of the first and second degrees will for the most part be considered.

EXERCISES

1. Verify Art. 38 by first finding the coördinates of the points of intersection of the loci of equations (1) and (2), and then substituting these coördinates in equation (3).

2. Find the equation of a curve that passes through all the points in which the following pairs of curves intersect:

$$(\alpha)\ \begin{Bmatrix} x^2+y^2=4, \\ x^2+2x+y^2=0. \end{Bmatrix} \quad (\beta)\ \begin{Bmatrix} y=2\sin x, \\ y=\cos x. \end{Bmatrix} \quad (\gamma)\ \begin{Bmatrix} x^2=4y, \\ y^2=4x. \end{Bmatrix}$$

EXAMPLES ON CHAPTER III

1. Are the points $(^-3, 17)$, $(4, 6)$, and $(5, 5)$ on the locus of $3x+2y=25$?

2. Is the point $\left(\frac{a}{3}, \frac{a}{2}\right)$ on the locus of $9x^2+4y^2=2a^2$?

3. The ordinate of a certain point on the locus of $x^2+y^2=25$ is 3; what is its abscissa? What is the ordinate if the abscissa is a?

Find by the method of Art. 36 where the following loci cut the axes of x and y:

4. $y=(x+2)(x+3)$.

5. $16x^2-9y^2=144$.

6. $x^2+6x+y^2=4y+3$.

7. The two loci $\frac{x^2}{4}-\frac{y^2}{9}=1$, $\frac{x^2}{4}+\frac{y^2}{9}=4$ intersect in four points; find the lengths of the sides and of the diagonals of the quadrilateral formed by these points.

8. A triangle is formed by the points of intersection of the loci of $x+y=a$, $x-2y=4a$, and $y-x+7a=0$. Find its area.

9. Find the distance between the points of intersection of the curves $3x-2y+6=0$, and $x^2+y^2=9$.

10. Does the locus of $y^2=5x$ intersect the locus of $9x-6y+5=0$?

11. Does the locus of $x^2-4y+4=0$ cut the locus of $x^2+y^2=1$?

12. For what values of m will the curves $x^2+y^2=9$ and $y=6x+m$ not intersect? [Cf. Art. 6]. Trace these curves.

13. For what value of b will the curves $y^2=4x$ and $y=x+b$ intersect in two distinct points? in two coincident points? in two imaginary points (*i.e.*, not intersect)?

14. Find those two values of c for which the points of intersection of the curves $y=3x+c$ and $x^2+y^2=36$ are coincident.

15. Find the equation of a curve that passes through all the points of intersection of $x^2+y^2=25$ and $y^2=4x$. Test the correctness of the result by finding the coördinates of the points of intersection and substituting them in the equation just found.

16. Write an equation that shall represent the combined loci of (1), (2), and (3) of Art. 30.

Discuss and construct the loci of the equations:

17. $(x^2-y^2)(y-\sin x)=0$.

18. $x^3-y^3=0$.

19. $x^4-y^4=0$.

20. $y=4x^3$.

21. $y^3=x^2$.

22. $y=10^x$.

23. Show that the following pairs of curves intersect each other in two coincident points; *i.e.*, are tangent to each other:

$$(\alpha)\quad y^2-8x-6y-63=0,\qquad 2y-x=23.$$

$$(\beta)\quad 9x^2-4y^2+54x-16y+29=0,\qquad 15x-8y+11=0.$$

24. Find the points of intersection of the curves

$$\frac{x^2}{9}+\frac{y^2}{25}=1 \quad \text{and} \quad \frac{x^2}{9}-\frac{y^2}{25}=1.$$

25. Find the intersection of $y=a^x$ and $x=0$. Plot.

26. Find the equations of the sides of the triangle whose vertices are the points $(2, 3)$, $(4, {}^-5)$, $(3, {}^-6)$. [Cf. Art. 32]. Test the resulting equations by substitution of the given coördinates.

27. Find the equations of the sides of the square whose vertices are $({}^-1, 0)$, $(1, 2)$, $(3, 0)$, $(1, {}^-2)$. Compare the equations of the parallel sides; of perpendicular sides.

28. Find the coördinates of the center of the square in Ex. 27. Then find the radius of the circumscribed circle, and (Art. 34) the equation of that circle. Test the result by finding the coördinates of the points of intersection of one of the sides with the circle. [Art. 36].

29. Find the equation of the path traced by a point which is always equidistant from the points

(α) $(4, 0)$ and $(0, {}^-4)$; (β) $(6, 4)$ and $(3, 3)$;

(γ) $(a+b, a-b)$ and $(a-b, a+b)$.

30. A point moves so that its abscissa always exceeds $\frac{2}{3}$ of its ordinate by 6. Find the equation of its locus, and trace the curve. Which side is positive? [Note, p. 44.]

31. A point moves so that the square of its abscissa is always 4 times its ordinate. Find the equation of its locus and trace the curve.

32. Find the equation of the locus of a point which moves so that the sum of its distances from the points $({}^-1, 3)$ and $(7, 3)$ is always 10. Trace and discuss the curve.

33. Find the equation of the locus of the point in Ex. 32, if the difference of its distances from the fixed points is always 2.

34. Express by a single equation the fact that a point moves so that its distance from the y-axis is always numerically 3 times its distance from the x-axis.

35. A point moves so that the square of its distance from the point $(a, 0)$ is 4 times its ordinate. Find the equation of its locus, and trace the curve.

36. A point moves so that its distance from the y-axis is $\frac{1}{2}$ of its distance from the origin. Find the equation of its locus, and trace the curve. Which side is positive?

37. A point moves so that the difference of the squares of its distances from the points $(1, 3)$ and $(4, 2)$ is 5. Find the equation of its locus and trace the curve.

38. Solve Ex. 37 if the word "sum" is substituted for "difference."

39. Let $A \equiv (a, 0)$, $B \equiv (b, 0)$, and $A' \equiv (^-a, 0)$ be three fixed points; find the equation of the locus of the point $P \equiv (x, y)$ which moves so that $\overline{PB}^2 + \overline{PA}^2 = 2\,\overline{PA'}^2$.

40. A point moves so that $\frac{1}{4}$ of its ordinate exceeds $\frac{1}{5}$ of its abscissa by 1. Find the equation of its locus and trace the curve.

41. Find the equation of the locus of a point that is always equidistant from the points $(3, ^-4)$ and $(5, 3)$; from the points $(3, ^-4)$ and $(2, 0)$. By means of these two equations find the coördinates of the point that is equidistant from the three given points.

42. Let $A \equiv (^-1, 3)$, $B \equiv (^-3, ^-3)$, $C \equiv (1, 2)$, $D \equiv (2, 2)$ be four fixed points, and let $P \equiv (x, y)$ be a point that moves subject to the condition that the triangles PAB and PCD are always equal in area; find the equation of the locus of P.

43. If the area of a triangle is 25 and two of its vertices are $(5, ^-6)$ and $(^-3, 4)$, find the equation of the locus of the third vertex.

44. Find the equation of the straight line through the points $(1, 7)$ and $(6, 11)$, using the fact that three points of a straight line determine a triangle whose area is zero.

The equation of a locus is the algebraic relation between the co-ordinates of x & y of every point on the locus.

CHAPTER IV

THE STRAIGHT LINE. EQUATION OF FIRST DEGREE

$$Ax + By + C = 0$$

40. In Chapter III it was shown that to every equation between two variables there corresponds a definite geometric locus, and that if the geometric locus be given, its equation may be found. It will now be helpful to consider some of the more elementary loci and their equations, and to apply analytic methods to the study of the properties of these curves. Since the straight line is a simple locus, and one whose properties are already well understood by the student, its equation will be examined first.

In studying the straight line, as well as the circle and other second degree curves, to be taken up in later chapters, it will be found best first to obtain the simplest equation which represents the locus, and to study the properties of the curve from that simple or *standard* equation; then to find methods for reducing to this standard form any other equation that represents the same locus.

41. Equation of straight line through two given points. The method of the present article is precisely the same as that illustrated by a numerical example in Art. 32.

Let two given fixed points be $P_1 \equiv (x_1, y_1)$, and $P_2 \equiv (x_2, y_2)$, and let $P \equiv (x, y)$ be *any* other point on the straight line through P_1 and P_2.

By definition of 'straight' line, the slope of the line remains constant; hence

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}, \qquad \text{[Art. 20]}$$

which may be written

N.B.

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}, \qquad [8]$$

Since $P \equiv (x, y)$ is *any* point on the line through P_1 and P_2, therefore equation [8] is satisfied by the coördinates of *every*

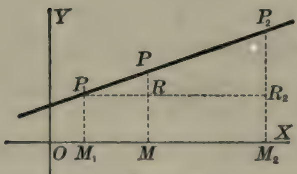


FIG. 28 *a*.

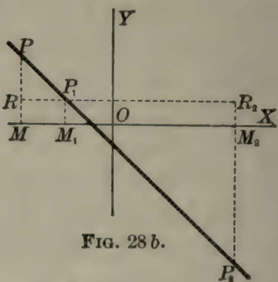


FIG. 28 *b*.

point on this line. That equation [8] is not satisfied by the coördinates of any point except such as are on the line P_1P_2 may be proved as was done in Art. 32.

Equation [8] then fulfills both requirements of the definition in (1) of Art. 28 and is therefore the equation of the straight line through the two points (x_1, y_1) and (x_2, y_2). This equation will be frequently needed and will be referred to as a *standard form;* it should be committed to memory.

NOTE. Since eq. [8] states a property of similar triangles which is not dependent upon the angle XOY, this equation is true for oblique axes.

42. Equation of straight line in terms of the intercepts which it makes on the coördinate axes. If the two given points in Art. 41

are those in which the line cuts the axes of coördinates, *i.e.*, $A \equiv (a, 0)$ and $B \equiv (0, b)$ (Fig. 29 *a*), then equation [8] becomes

$$\frac{y-0}{b-0} = \frac{x-a}{0-a};$$

N. B.

that is, $$\frac{x}{a} + \frac{y}{b} = 1, \qquad [9]$$

where a and b are the intercepts which the line cuts from the axes (where it is assumed $a \neq 0$, $b \neq 0$).

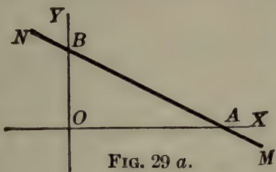


FIG. 29 *a*.

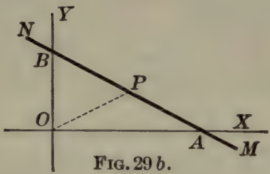


FIG. 29 *b*.

This is another standard form of the equation of the straight line; it is known as the **symmetrical** or the **intercept** form.

NOTE. In equation [9] both x and y are variables, but are not independent; each is an implicit function of the other. For any particular line a and b are constants, but they may represent other constants in the equation of another line, *i.e.*, they are arbitrary constants, and are often called **parameters** of the line.

EXERCISES

1. Show that equation [9] is not satisfied by the coördinates of any point except those lying on MN.

2. Write down the equations of the lines through the following pairs of points:

(α) $(3, 4)$ and $(4, 2)$; (γ) $(^-6, 1)$ and $(^-2, ^-5)$;

(β) $(3, 4)$ and $(4, ^-2)$; (δ) $(^-15, ^-3)$ and $\left(\frac{8}{3}, \frac{^-7}{9}\right)$.

3. Write the equations of the lines which make the following intercepts on the x- and y-axes respectively.

(α) 7 and 4; (γ) $-\frac{1}{2}$ and $+\frac{4}{3}$;

(β) -6 and 10; (δ) $-3a$ and $-\frac{a}{2}$.

4. What do equations [8] and [9] become if one of the given points is the origin?

5. By drawing, in Fig. 29 *b*, a perpendicular PQ from P to the x-axis, derive equation [9] from the similar triangles QAP and OAB.

6. Is equation [9] true if P is on MN but not between A and B in Fig. 29 *b*?

7. Are equations [8] and [9] true if the coördinate axes are not at right angles to each other? Show by similar triangles.

8. Is the point (3, 3) on the line through the points (2, 3) and (5, 7)? On which side of this line is it? Which is the negative side of this line?

9. What intercepts does the line through the points $(^-6, 1)$ and $(5, ^-3)$ make on the axes?

10. The vertices of a triangle are: $(8, ^-10)$, $(4, 6)$, and $(6, ^-12)$. Find the equations of the sides; also of the three medians; then find the coördinates of the point of intersection of two of these medians, and show that these coördinates satisfy the equation of the other median. What proposition of plane geometry is thus proved?

11. Find the tangent of the angle [the "slope," cf. Art. 20] which the line in Ex. 9 makes with the x-axis.

12. Draw the line whose equation is $\frac{x}{3}+\frac{y}{2}=1$, and then find the equations of the two lines which pass through the origin and trisect that portion of this line which lies in the first quadrant.

43. Equation of straight line through a given point and in a given direction. Let $P_1 \equiv (x_1, y_1)$ be the given point, and let the direction of the line be given by the angle $XAP = \theta$ which the line makes with the x-axis; also let $P \equiv (x, y)$ be *any* point on the given line and denote the slope, *i.e.*, $\tan\theta$, by m.

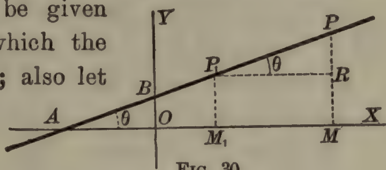


FIG. 30.

Using again the fact that the slope of the line is constant,

N B

$$m = \frac{y - y_1}{x - x_1}, \qquad \text{[Art. 20]}$$

that is,

$$\boldsymbol{y - y_1 = m(x - x_1)}, \qquad \textbf{[10]}$$

which is the desired equation.

Corollary. If the given point be $B \equiv (0, b)$, *i.e.*, the point in which the line meets the y-axis, then equation [10] becomes

N B

$$\boldsymbol{y = mx + b}. \qquad \textbf{[11]}$$

Equation [11] is usually spoken of as the **slope** form of the equation of the straight line.

EXERCISES

1. What do the constants m and b in equation [11] mean? Draw the line for which $m = 3$ and $b = 4$, also that for which $m = -1$ and $b = -\frac{3}{2}$.

2. What is the effect on the line represented by eq. [11] if b is changed while m remains the same? What if m be changed and b left unchanged?

3. Describe the effect on the line [10] of changing m while x_1 and y_1 remain the same; also the effect resulting from a change in x_1 while m and y_1 remain the same.

4. Write the equation of a line through the point $(^-3, 7)$, and making with the x-axis an angle of $30°$; of $^-60°$; of $\left(\frac{2\pi}{3}\right)^{(r)}$; of $\left(-\frac{7\pi}{6}\right)^{(r)}$.

5. Write the equations of the following lines:

(α) slope 4, y-intercept 4; (β) slope $\frac{1}{2}$, y-intercept $^-3$;
(γ) slope $^-2$, x-intercept $^-\frac{3}{4}$.

6. A line has the slope $^-6$; what is its y-intercept if it passes through the point $(7, 1)$?

7. What must be the slope of a line whose y-intercept is $^+3$, in order that it may pass through the point $(^-5, 5)$?

8. Is the point $(1, \frac{5}{2})$ on the line passing through the point $(0, ^-3)$, and making an angle $\tan^{-1}\frac{11}{2}$ with the x-axis?

9. How do the lines $y=3x-1$, $y=3x+7$, and $ky-3kx+15=0$ differ from each other? What have they in common? Draw these lines.

10. What is common to the lines $y=3x-1$, $2y=5x-2$, and $7x-3y=3$?

11. What is the slope of line [8], p. 60? of line [9], p. 61?

12. Derive equation [11], p. 63, independently of equation [10], p. 63.

44. Equation of straight line in terms of the perpendicular from the origin upon it and the angle which that perpendicular makes

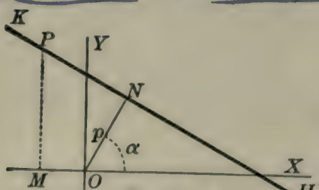


FIG. 31 *a*.

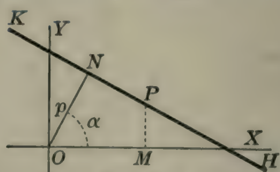


FIG. 31 *b*.

with the x-axis. Let KH be the line whose equation is sought and let the perpendicular $(ON=p)$ from O upon this line,

and the angle (α) which this perpendicular makes with the x-axis, be given. Also let $P \equiv (x, y)$ be any point on KH; then by projection upon ON [Art. 13],

$$OM \cos(\alpha) + MP \cos\left(\frac{\pi}{2} - \alpha\right) = ON,$$

i.e., $$\boldsymbol{x \cos \alpha + y \sin \alpha = p}, \qquad \textbf{[12]}$$

which is the required equation.

Equation [12] is known as the normal form of the equation of the straight line.

In the following pages α will, unless otherwise stated,* be regarded as positive, and less than 180°.

Corollary. If β is the angle to p *from* the y-axis, equation [12] takes the symmetrical form $x \cos \alpha + y \cos \beta = p$. Similarly, if α and β are respectively the angles which the line makes with the x and y axes, equation [10] takes the symmetrical form

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta}.$$

[Compare Arts. 148 and 151].

EXERCISES

1. The perpendicular from the origin upon a certain line is 6; this perpendicular makes an angle of $\frac{\pi}{6}$ with the x-axis; what is the equation of the line?

2. If in equation [12] p is increased while α remains the same, what is the effect upon the line? If α be changed while p remains the same, what is the effect?

3. A certain line is 4 units distant from the origin, and makes an angle of 120° with the x-axis; what is its equation?

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* Cf. Art. 51.

4. Given $\alpha = 45°$, what must be the length of p in order that the line HK (see Fig. 31 a) shall pass through the point (2, 7)?

5. A line passes through the point ($^-4$, $^-3$), and a perpendicular upon it from the origin makes an angle of 225° with the x-axis. What is the equation of this line?

6. A line passes through the origin with the slope $m = 1$. What is the value of α? of p?

7. If two parallel lines are on opposite sides of the origin do they have the same angle α? Explain.

8. State the *parameters* [cf. note, Art. 42] for each of the standard equations of the straight line.

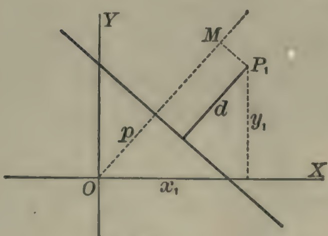

FIG. 32.

9. Show by projection that the distance d of $P_1 \equiv (x_1, y_1)$ from the line

$$x \cos \alpha + y \sin \alpha = p$$

is $d = x_1 \cos \alpha + y_1 \sin \alpha - p$.

45. Summary. The results of Arts. 40 to 44 may be briefly summarized as follows:

The position of a straight line is determined by

I. Two points through which it passes. Eq. [8], p. 60

Special case:

one point on the x-axis, the other upon the y-axis. Eq. [9], p. 61

II. One point and its direction. Eq. [10], p. 63

Special cases:

(1) the given point on the y-axis, Eq. [11], p. 63

(2) the point given by its distance and direction from the origin, and the direction of the line being given by the perpendicular to it from the origin. Eq. [12], p. 65

Of the five standard forms of the equation of the straight line, [8] and [9] are independent of this angle between the coördinate axes,* while [10], [11], and [12] (m, α, and p retaining their present meanings) are true only when the axes are rectangular. It may also be pointed out that, from the nature of its derivation, equation [8] is inapplicable when the line is parallel to either axis; equation [9] is inapplicable when the line passes through the origin; and equations [10] and [11] are not applicable when the line is parallel to the y-axis.

46. Reduction of the general equation $Ax + By + C = 0$ to the standard forms. Determination of a, b, m, p, and α in terms of A, B, and C.

The most general equation of the first degree in two variables may be written in the form

$$Ax + By + C = 0$$

where A, B, and C are constants, with A positive, and neither A nor B is zero.†

(1) *Reduction to the standard form* $\frac{x}{a} + \frac{y}{b} = 1$ (*symmetric or intercept form*).

The equation $$Ax + By + C = 0 \qquad (1)$$

represents *some* locus which is not changed by multiplication by a constant, and transposition [Art. 35]; therefore

N. B

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \ddagger \qquad (2)$$

* See Art. 41, Note.

† If either A or B, say A, is zero, then the equation may be written in the form : $y = -\frac{C}{B}$, which is the equation of a straight line parallel to the x-axis, and at the distance $-\frac{C}{B}$ from it [cf. Art. 27, (2)].

‡ Where $C \neq 0$. See Art. 42.

represents the same locus. But equation [2] is in the standard form for a straight line [Art. 42], and the intercepts are:

$$a = -\frac{C}{A}, \text{ and } b = -\frac{C}{B}.$$

These intercepts are in practice readily found by substituting $y = 0$, and $x = 0$, respectively, in the given equation.

Since this demonstration does not depend upon the angle between the axes, therefore it applies whether the axes are oblique or rectangular; hence the theorem:

Every equation of the first degree between two variables, when interpreted in Cartesian coördinates, represents a straight line.

Because of this fact, such an equation is often spoken of as a **linear** equation.

NOTE. In the equation $Ax + By + C = 0$, there are apparently three constants; in reality, there are but two independent constants, viz. the ratios of the coefficients. [Cf. Art. 35]. This corresponds to the fact that a straight line is determined geometrically by two conditions. The equation may be written in the form $Ax + By + 1 = 0$, wherein A and B may be fractions.

(2) *Reduction to the standard form $y = mx + b$ (slope form).*

The equation $Ax + By + C = 0$ has the same locus as has the equation

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)^{*} \tag{3}$$

N. B

[see Art. 35]; hence, by comparison with equation (11)

$$m = -\frac{A}{B}, \text{ and } b = -\frac{C}{B};$$

and equation (3) is of the required form.

*If $B = 0$, the line represented by equation [1] is parallel to the y-axis, and the slope form of the equation is inapplicable [Art. 45], in that case, this reduction also fails.

(3) *Reduction to the standard form $x\cos\alpha + y\sin\alpha = p$ (normal form).*

If any two numbers A and B be taken as the lengths of the legs of a right triangle, then the hypotenuse makes with the leg A an angle α such that

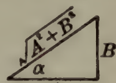

FIG. 33.

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}} \text{ and } \sin\alpha = \frac{B}{\sqrt{A^2+B^2}}.$$

Now, divide equation [1] through by $\sqrt{A^2+B^2}$; and transpose the constant term:

IV, B

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{-C}{\sqrt{A^2+B^2}}. \tag{4}$$

This is therefore of the form required,

$$\text{and} \quad p = \frac{-C}{\sqrt{A^2+B^2}}, \ \cos\alpha = \frac{A}{\sqrt{A^2+B^2}}, \ \sin\alpha = \frac{B}{\sqrt{A^2+B^2}}.$$

Since α is to be taken positive, and less than 180° [Art. 44], the sign of $\sqrt{A^2+B^2}$ is to be taken the same as that of B in the given equation.

47. To trace the locus of an equation of the first degree. Since the locus of an equation of the first degree in two variables is a straight line, and a straight line is determined by two of its points, therefore to trace the locus it is necessary to find only the two points most easily determined, viz. those points where the line cuts the axes, respectively. If the line is parallel to an axis only one point is needed.

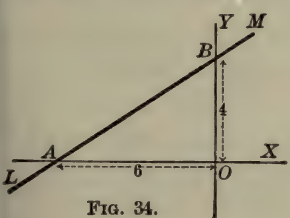

FIG. 34.

E.g. The locus of the equation

$$2x - 3y + 12 = 0$$

passes through the two points (0, 4) and ($^-6$, 0); therefore LM is the locus.

EXERCISES

1. Reduce the following equations to the intercept (symmetric) form, and draw the lines which they represent:

(α) $2x - 3y + 12 = 0$; (β) $6x - 4y + 1 = 10x + 3$;

(γ) $2y = 15 - y + 5x$; (δ) $\dfrac{x - 2y + 1}{3 + 7y} = 9$.

2. Reduce to the slope form, and then trace the loci:

(α) $7x - 5y + 6(y - 3x) = -10x + 4$; ($\beta$) $3x + 2y + 6 = 0$;

(γ) $x - 5 = 3 - 3y$.

Which is the positive side of the line (β)?

3. Reduce to the normal form, and then trace the loci:

(α) $4x + 3y = 15$; (β) $4x - 3y + 15 = 0$;

(γ) $x - 3y = 5 + 6x$; (δ) $\frac{2}{3}x = y - 5$.

4. Show that the lines $9x + 5 = 3y$ and $3x - y = 81$ are parallel.

5. What is the slope of the line between the two points (3, $^-1$) and (2, 2)? What is its distance from the origin?

6. A line passes through the point (6, 5) and has its intercepts on the axes equal and both positive. Find its equation and its distance from the origin.

7. A straight line passes through the point ($^-2$, 1) and is such that the portion of it between the axes is bisected by that point. What is the slope of the line?

8. What are the intercepts which the line through the points $(3, {}^-1)$ and $(7, 6)$ makes on the axes? Through the points $({}^-a, {}^-2a)$ and $({}^-b, {}^-2b)$?

9. What system of lines obtained by varying the parameter b is represented by the equation $y = 8x + b$?

10. What system of lines obtained by varying the parameter m is represented by the equation $y = mx + 10$?

11. What family (system) of lines obtained by varying the parameter α is represented by the equation $x \cos \alpha + y \sin \alpha = 10$? To what curve is each line of the family tangent?

12. Find $\cos \alpha$ and $\sin \alpha$ for the lines

(α) $y = mx + b$; (β) $\frac{x}{a} - \frac{y}{b} = 1$;

(γ) $\frac{2}{x} = \frac{3}{y}$; ($\delta$) $7x - 5y + 10 = 0$.

13. Find by means of $\cos \alpha$ and $\sin \alpha$ what quadrant is crossed by each of the lines:

(α) $2x + 2 = 3y$; (β) $3x + 5y + 15 = 0$; (γ) $x + \sqrt{3}y - 10 = 0$.

14. What must be the slope of the line $+kx - 4y = -17$ in order that it shall pass through the point $(1, 3)$? Has k a finite value for which this line will pass through the origin?

15. Determine the values A, B, C in order that the line

$$Ax + By + C = 0$$

shall pass through the points $(0, 3)$ and $(-12, 0)$. [Art. 46, Note].

16. Derive equation [8] by supposing (x_1, y_1) and (x_2, y_2) to be two points on the line $y = mx + b$; and thence finding values for m and b.

17. Find the slopes of the lines $5y - 3x = 9$ and $3y + 5x - 10 = 0$; and thence show that these lines are perpendicular to each other.

18. Find $\cos\alpha$ for each of the lines $9x+y-9=0$ and $x-9y+2=0$, and then show that the two lines are perpendicular to each other.

19. Show by means of: (1) the slopes; (2) the angles; that the lines

$$2y-3x=7,\qquad 4y-6x+5=0,\qquad 10y-15x+c=0$$

are all parallel.

20. Reduce the equation $Ax+By+C=0$ to the normal form, *i.e.*, to the form $x\cos\alpha+y\sin\alpha=p$. Suggestion: the two equations, as representing the same line, make the same intercept on the axes.

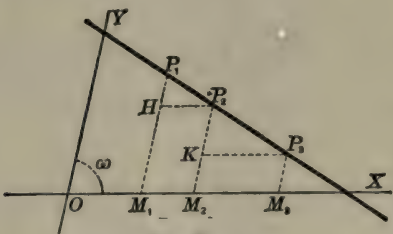

FIG. 35.

21. Prove that the locus of $Ax+By+C=0$ is a straight line, (*a*) by using similar triangles P_1HP_2 and P_2KP_3, thus showing that the sum of the angles at P_2 is equal to 2 rt. ∠s. (*b*) by using Article 24.

48. To find the angle made by one straight line with another. Let the equations of the lines be

$$y=m_1x+b_1, \qquad (1)$$

and $$y=m_2x+b_2, \qquad (2)$$

where $m_1=\tan\theta_1$, $m_2=\tan\theta_2$, and θ_1, θ_2 are the angles which these lines make, respectively, with the x-axis. It is required to find the angle ϕ, measured from line (1) to line (2).

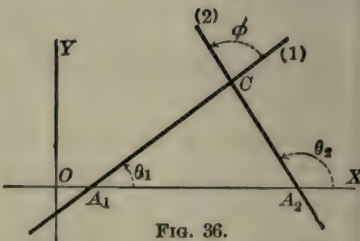

FIG. 36.

Since $$\phi = \theta_2 - \theta_1,$$

$$\therefore \quad \tan\phi = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \cdot \tan\theta_2}, \qquad \text{[Art. 12]}$$

N.B.

i.e., $$\tan\phi = \frac{m_2 - m_1}{1 + m_1 m_2}. \qquad [13]$$

If the angle were measured from line (2) to line (1), it would be the negative, or else the supplement, of ϕ; in either case its tangent would be the negative of that given by formula [13].

If the equations of the lines be given in the form:

$$A_1x + B_1y + C_1 = 0, \qquad (3)$$

and $$A_2x + B_2y + C_2 = 0, \qquad (4)$$

then formula [13] becomes

$$\tan\phi = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}. \qquad [14]$$

EXERCISES

Find the tangent of the angle from the first line to the second in each of the following cases, and draw the figures:

1. $2x - 1y - 7 = 0, \quad 3x - 4y - 3 = 0;$
2. $x - 2y + 1 = 0, \quad 5x + 12y + 6 = 0;$
3. $4x = 6y + 9, \quad 3y = 2x + 2;$
4. $\frac{x}{a} - \frac{y}{b} = 1, \quad \frac{x}{a} + \frac{y}{b} = 1;$
5. $x\cos\alpha + y\sin\alpha = p, \quad \frac{x}{a} + \frac{y}{b} = 1.$

49. Condition that two lines are parallel or perpendicular. From formula [13] can be seen at once the relations that were shown in Art. 21 to hold for parallel and perpendicular lines.

For if lines (1) and (2), Art. 48, are parallel, then $\phi=0$, $\tan\phi=0$, and
$$m_1=m_2.\text{*}$$
If lines (1) and (2) are perpendicular to each other, then $\phi=\frac{\pi}{2}$, $\tan\phi=\infty$, and $\quad m_1=-\frac{1}{m_2}.$

Similarly, if the lines be given by equations in the forms (3) and (4), Art. 48, then from equation 14, if the lines are parallel,
$$\frac{A_1}{B_1}=\frac{A_2}{B_2};$$
if the lines are perpendicular,
$$\frac{A_1}{B_1}=-\frac{B_2}{A_2}.$$

The conditions just found enable one to write down readily the equations of lines which are parallel or perpendicular to given lines, and which also pass through given points.

E.g., a line parallel to the line
$$y=3x+7 \tag{1}$$
must have the same slope, $m=3$, and hence must have its equation of the form
$$y=3x+b. \tag{2}$$
Now b may be determined by a second condition, *e.g.*, by the requirement that the line shall pass through the point (1, 5). Then the coördinates must satisfy equation (2);

i.e., $\quad 5=3\cdot 1+b, \quad \therefore b=2,$

and
$$y=3x+2, \tag{3}$$

---

* It must not be forgotten that this conclusion is drawn only for lines that are *not* perpendicular to the x-axis; because if the lines are perpendicular to the x-axis then equations (1) and (2) are inapplicable. [Cf. Art. 45].

represents a line parallel to line (1) and through the point (1, 5).

Similarly, $$y = -\tfrac{1}{3}x + b \tag{4}$$

is the equation of a line perpendicular to line (1), for all values of b. And again, for all values of K, the lines

$$3x + 5y - 15 = 0 \quad \text{and} \quad 3x + 5y + K = 0$$

are parallel; while the lines

$$3x + 5y - 15 = 0 \quad \text{and} \quad 5x - 3y + K' = 0$$

are perpendicular. Here also the values of K and K' may be so determined that the respective lines shall pass through any given point.

This condition for parallelism and for perpendicularity of two lines may also be stated thus: *two lines are parallel if their equations differ* (or may be made to differ) *only in their constant terms; two lines are perpendicular if the coefficients of x and y in the one are equal* (or can be made equal), *respectively, to the coefficients of $-y$ and x in the other.*

EXERCISES

1. Write down the equations of the set of lines parallel to the lines: (α) $y = 6x - 2$; (β) $3x - 7y = 3$;

(γ) $x\cos 30° + y \sin 30° = 8$; (δ) $\dfrac{x}{a} - \dfrac{y}{b} = 1$.

2. Explain why it is that the constant term in the answers to Ex. 1 is left undetermined or arbitrary.

3. Find the tangent of the angle between the lines (α) and (β) in Ex. 1; also for the lines (β) and (δ), and (α) and (δ) of Ex. 1.

4. Write the equations of lines perpendicular to those given in Ex. 1.

5. By the method of Art. 49 find the equation of the line that passes through the point $(^-9, 1)$, and is parallel to the line $y = 6x - 2$.

6. Solve Ex. 5 by means of equation [10], Art. 43.

7. Find the equation of the line that is parallel to the line $Ax + By + C = 0$ and that passes through the point (x_1, y_1); make two solutions, one by the method of Ex. 6, and the other by Ex. 5.

Find the equation of the straight line:

8. Through the point $(2, {}^-5)$ and parallel to the line $2y = x - 10$.

9. Through the point $(^-1, {}^-1)$ and perpendicular to $2y = 2x - 10$; solve by two methods.

10. Through the point $(1, 1)$ and parallel to the line

$$\frac{3}{2}x - \frac{7}{5}y = \frac{x - y + 1}{9}.$$

11. Perpendicular to the line $7y + 2x - 1 = 0$, and passing through the point midway between the two points in which this line meets the coördinate axes.

12. Find the foot of the perpendicular from the origin to the line $7x - 5y = 10$.

50. Line which makes a given angle with a given line. The formula

$$\tan\phi = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} \qquad \text{[Art. 48]}$$

states the relation existing between the tangents of the angles θ_1, θ_2, and ϕ (see Fig. 36), hence if any two of these angles are known, this equation determines the value of the third. Thus

this formula may be employed to determine the slope of a line that shall make a given angle with a given line.

E.g., to find the equation of a line that shall make the angle 60° with the line $3y - 5x + 7 = 0$. Here $\phi = 60°$, $\therefore \tan\phi = \sqrt{3}$, and $m_1 = \frac{5}{3}$. Since in equation [13] ϕ is measured *from* line (1), substituting in equation [13],

$$\sqrt{3} = \frac{\tan\theta_2 - \frac{5}{3}}{1 + \frac{5}{3}\tan\theta_2}, \quad \text{whence} \quad \tan\theta_2 = \frac{5 + 3\sqrt{3}}{3 - 5\sqrt{3}};$$

and
$$y = \frac{5 + 3\sqrt{3}}{3 - 5\sqrt{3}} \cdot x + k$$

is the equation of a line fulfilling the required conditions; k may be so determined that this line shall also pass through any given point.

It is to be remarked that through any given point there may be drawn *two* lines, each of which shall make, with a given line, an angle of any desired magnitude.

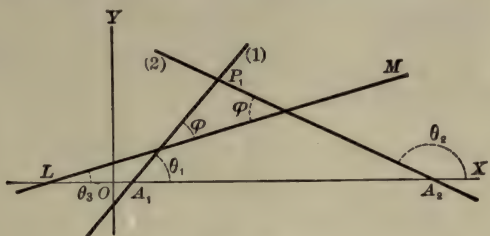

Fig. 37.

Thus, through $P_1 \equiv (x_1, y_1)$ the lines (1) and (2) may be so drawn that each shall make an angle ϕ with the given line LM: but line (1) makes the angle $+\phi$, while line (2) makes the angle $-\phi$; which must be remembered in applying formula [13].

EXERCISES

1. Find the equations of the two lines which pass through the point (8, 5), and each of which makes an angle of 45° with the line $3x - 2y = 6$.

2. Show that the equations of the two straight lines passing through the point (3, 5) and inclined at 45° to the line $2x - 3y - 7 = 0$ are

$$5x - y - 10 = 0 \quad \text{and} \quad x + 5y - 28 = 0.$$

Find the equation of the straight line:

3. Making an angle of $+\frac{\pi}{4}$ with the line $4x - 3y = 7$; construct the figure. Why is there an undetermined constant in the resulting equation?

4. Making an angle of $-60°$ with the line $5x + 12y + 1 = 0$; construct the figure.

5. Making an angle of $+30°$ with the line $x - 2y + 1 = 0$, and passing through the point (1, 3); making an angle of $-30°$, and passing through the same point.

6. Making an angle of $\pm 135°$ with the line $x - y = 2$, and passing through the origin.

7. Making the angle $\tan^{-1}\left(+\frac{b}{a}\right)$ with the line $\frac{x}{a} - \frac{y}{b} = 1$, and passing through the point $\left(\frac{a}{2}, -\frac{b}{2}\right)$.

8. Find the equation of a line through the point (4, 5) forming with the lines $2x - y + 3 = 0$ and $3y + 6x = 7$ a right-angled triangle. Find the vertices of the triangle (two solutions).

9. Show that the triangle whose vertices are the points $(4, 2)$, $(6, {}^-4)$, $({}^-8, {}^-2)$ is a right triangle.

10. Prove analytically that the perpendiculars erected at the middle points of the sides of the triangle, the equations of whose sides are

$$x-1=0,\quad y-1=0,\quad \text{and}\quad x+y=10,$$

meet in a point which is equidistant from the vertices.

11. Find the equations of the lines through the vertices and perpendicular to the opposite sides of the triangle in exercise 10. Prove that these lines also meet in a common point.

12. A line passes through the point $(^-2, 3)$ and is parallel to the line through the two points $(4, 7)$ and $(^-1, ^-9)$; find its equation.

13. Find the equation of the line which passes through the point of intersection of the two lines $10x+5y+11=0$, and $x+2y+14=0$, and which is perpendicular to the line $x+7y+1=0$.

This problem may be solved by first finding the point of intersection $(\frac{16}{5}, -\frac{43}{5})$ of the two given lines, and then, by formula [10], p. 63 [see also Art. 49], writing the equation of the required line, viz.:

$$y+\tfrac{43}{5}=7(x-\tfrac{16}{5}),$$

which reduces to $\quad 7x-y=31.$

Or, it may also be solved somewhat more briefly, and much more elegantly, by employing the theorem of Art. 38. By this theorem the equation of the required line is of the form

$$10x+5y+11+k(x+2y+14)=0,$$

i.e., $\quad (10+k)x+(5+2k)y+11+14k=0.$

It only remains to determine the constant k, so that this line shall be perpendicular to $x+7y+1=0.$

By Art. 49 its slope must be $-\frac{1}{-\frac{1}{7}}=7$, hence $-\frac{10+k}{5+2k}=7$, whence $k=-3$.

Substituting this value of k above, the required equation becomes $7x-y=31$, as before.

14. By the second method of exercise 13 find the equation of the line which passes through the point of intersection of the two lines $2x+y=5$ and $x=3y-8$, and which is: (1) parallel to the line $4y=6x+1$; (2) perpendicular to this line; (3) inclined at an angle of $60°$ to this line; (4) passes through the point $(^-1, 3)$.

15. Solve exercise 10 by the method of exercise 14.

16. Do the lines $4x-2y=3$, $3x-y+\frac{1}{2}=0$, and $5x-2y-1=0$ meet in a common point? What are the angles they make with each other?

17. Find the angles of the triangle of exercise 10.

18. When are the lines

$$x+(a+b)y+c=0 \quad \text{and} \quad a(x+ay)+b(x-by)+d=0$$

parallel? when perpendicular?

19. Find the value of p for each of the two parallel lines

$$y=4x+14 \quad \text{and} \quad y=4x-10;$$

and hence find the distance between these lines. [Cf. Art. 46 (3) and (4)].

20. What is the distance between the two parallel lines

$$3x-5y+6=0 \quad \text{and} \quad 10y-6x=7?$$

21. Find the cosine of the angle between the lines

$$4y-x+8=0 \quad \text{and} \quad 6y-x+9=0.$$

22. What relation exists between the two lines

$$y = 4x + 8 \quad \text{and} \quad y = -4x - 6?$$

23. Find the angle between the two straight lines $3x = 4y + 7$ and $5y = 12x + 6$; and also the equations of the two straight lines which pass through the point (4, 5) and make equal angles with the two given lines.

24. Find the angle between the two lines

$$x + 3y + 10 = 0 \quad \text{and} \quad 2x + y - 10 = 0.$$

Find also the coördinates of their point of intersection, and the equations of the lines drawn perpendicular to them from the point $(^-2, 3)$.

51. The distance of a given point from a given line. This problem is easily solved for any particular case, thus: Find the equation of the line which passes through the given point and which is parallel to the given line [Art. 49], then find the distances p_1 and p_2 from the origin to the given line and the parallel line, respectively [Art. 46, (3) and (4)], and finally subtract the former distance from the other; the result is the distance *from* the given line to the given point.*

E.g., find the distance d of the point $P_1 \equiv (2, \frac{3}{2})$ from the line

$$3x + 4y - 7 = 0. \qquad (1)$$

The parallel line

$$3x + 4y + k = 0$$

contains the point $(2, \frac{3}{2})$: hence

$$k = -6 - 6 = -12,$$

The equation becomes

$$3x + 4y - 12 = 0. \qquad (2)$$

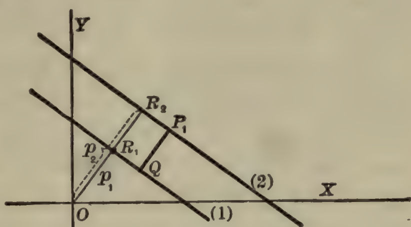

FIG. 38.

* Cf. Art. 44, Ex. 9.

Now for lines (1) and (2) the perpendiculars from the origin are respectively, by Art. 46 (3),

$$p_2 = \frac{+12}{+\sqrt{4^2+3^2}} = +\frac{12}{5}; \quad \text{and} \quad p_1 = \frac{7}{5}.$$

Hence, $$d = p_2 - p_1 = +1.$$

Similarly, in general, to find the distance of any given point $P_1 \equiv (x_1, y_1)$ from any given line,

$$Ax + By + C = 0. \tag{1}$$

The equation of the parallel line through P_1 is

$$Ax + By - (Ax_1 + By_1) = 0. \tag{2}$$

Therefore $$p_2 = \frac{Ax_1 + By_1}{\sqrt{A^2+B^2}}, \quad p_1 = \frac{-C}{\sqrt{A^2+B^2}},$$

wherein the sign of the radical is to be chosen in accord with Art. 46 (3);

hence $$\boldsymbol{d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}}. \tag{15}$$

If the equation of a given line is so written that its second member is zero, this formula may be translated into words thus: *To get the distance of a given point from a given line, write the first member of the equation alone, substitute for the variables therein the coördinates of the given point, and divide the result by the square root of the sum of the squares of the coefficients of x and y in the equation,*— the sign of this square root being chosen the same as that of the number represented by B.

In formula [15] the numerator is positive or negative according as P_1 is on the positive or negative side of the given line [see Art. 32, Note]; *i.e.*, for points on opposite sides of the given line the corresponding distances are opposite in sign. The origin is at a negative distance from the given line if B and C have opposite signs.

EXERCISES

1. Find the distance of the point $(2, {}^-7)$ from the line $3x - 6y + 1 = 0$.

By formula [15], $d = \dfrac{3 \cdot 2 - 6({}^-7) + 1}{-\sqrt{3^2 + 6^2}} = -\dfrac{49}{3\sqrt{5}}$.

The origin is on the positive side of the line $3x - 6y + 1 = 0$, but at a negative distance; hence the point $(2, {}^-7)$, also, is on the positive side of the line.

2. Find the distance of the point (5, 4) from the line $5y + 4x = 20$. On which side of the line is the point?

3. Find the distance of the point (2, 7) from the line $6y - 4x = 17$.

4. Find the distance of the point $({}^-a, {}^-b)$ from the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

5. Find the distance of the intersection of the two lines, $y + 4 = 3x$ and $5x = y - 2$, from the line $2y - 7 = 9x$. On which side of the latter line is the point?

6. Find the distance of the point of intersection of the lines $2x - 5y = 11$ and $4x = 3y + 15$ from the line $\dfrac{1}{2}x + \dfrac{y-5}{4} = 6$. On which side of the latter line is the point? Plot the figure.

7. How far is the point $({}^+6, {}^+1)$ from $3y = 7x + 8$? On which side?

8. By the method of Art. 51, find the distance of the origin from the line $2x - 5y = 7$; also from the line $Ax + By + C = 0$. Check the results by Art. 46 (3).

9. Find the distance of the point $({}^+4, {}^+5)$ from the line joining the two points $(3, {}^-1)$ and $({}^-4, 2)$. On which side is it?

10. Find the distance of the point (x_1, y_1) from the line $y = mx + b$.

11. Find the altitudes of the triangle formed by the lines whose equations are $3x+y+4=0$, $3x-5y+34=0$, and $3x-2y+1=0$. Check the result by finding the area of the triangle in two ways.

12. Show analytically that the locus of a point which moves so that the sum of its distances from two given straight lines is constant is itself a straight line.

13. Express by an equation that the point $P_1 \equiv (x_1, y_1)$ is equally distant from the two lines $x-2y=11$ and $3x=4y+5$. (Give two answers.) Should P_1 move in such a way as to be always equidistant from these two lines, what would be the equation of its locus?

14. Find, by the method of exercise 13, the equations of the bisectors of the angle formed by the lines $4x+3y=12$ and $3x+4y=24$.

52. Bisectors of the angles between two given lines. The bisector of an angle is the locus of a point which moves so that it is always equally distant (numerically) from the sides of the angle. From this property its equation may easily be found, as in Art. 51, exercises 13 and 14. The fact that in two opposite quarters formed by the intersecting lines the distances of the moving point from the given lines are *alike* in sign, while in the other two quarters the distances *differ* in sign [Art. 51] leads to the two equations for the two bisectors.

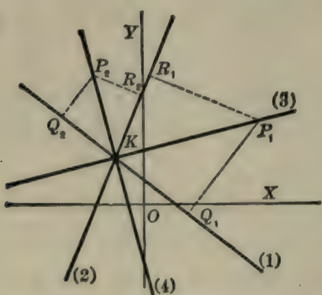

FIG. 39.

Fig. 39, $Q_1P_1 = -R_1P_1$

and $Q_2P_2 = +R_2P_2$.

The resulting equations will be found to show the well-known fact that the two bisectors are perpendicular to each other.

E.g., the bisectors of the angles, Fig. 39, between the lines

$$3x+4y-1=0 \quad \text{and} \quad 12x+5y+6=0$$

are given by the equations

$$\frac{3x+4y-1}{+5}=+\frac{12x+5y+6}{+13},$$

$$\frac{3x+4y-1}{+5}=-\frac{12x+5y+6}{+13};$$

i.e., by $21x-27y+43=0$ and $99x+77y+17=0$. The slopes are $m=\frac{7}{9}$ and $m'=-\frac{9}{7}$; hence the lines are perpendicular to each other.

EXERCISES

1. Find the equations of the bisectors of the angles between the two lines $x-y+6=0$ and $\dfrac{3x-4}{2}=5y-7$.

2. Show that the line $x+y+10=0$ bisects one of the angles between the two lines $4x-3y+17=0$, and $3x-4y+7=0$. Which angle is it? Find the equation of the bisector of the other angle.

3. Show analytically that the bisectors of the interior angles of the triangle whose vertices are the points (0, 0), (0, 3), and (4, 0) meet in a common point.

4. Show analytically, for the triangle of Ex. 3, that the bisectors of one interior and the two opposite exterior angles meet in a common point.

5. Find the angle from the line $3x+y+12=0$ to the line $ax+by+1=0$ and also the angle from the line $ax+by+1=0$ to the line $x+2y-1=0$.

By imposing upon a and b the two conditions: (1) that the angles just found are equal, and (2) that the line $ax + by + 1 = 0$ passes through the intersection of the other two lines, determine a and b so that this line shall be a bisector of one of the angles made by the other two given lines.

53. The equation of two lines. By the reasoning given in Art. 37 it is shown that if two straight lines are represented by the equations

$$A_1x + B_1y + C_1 = 0 \tag{1}$$

and

$$A_2x + B_2y + C_2 = 0, \tag{2}$$

then *both* these lines are represented by the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0; \tag{3}$$

i.e., two straight lines are here represented by an equation of the *second* degree.

Conversely, if an equation of the second degree, whose second member is zero, can have its first member separated into two first degree factors, with real coefficients, as in equation (3), then its locus consists of two straight lines.

Thus the equation

$$2x^2 - xy - 3y^2 + 9x + 4y + 7 = 0$$

may be written in the form

$$(2x - 3y + 7)(x + y + 1) = 0,$$

which shows that it is satisfied when $2x - 3y + 7 = 0$, and also when $x + y + 1 = 0$. Its locus is therefore composed of the two lines whose equations are:

$$2x - 3y + 7 = 0, \text{ and } x + y + 1 = 0.$$

The test to be satisfied in order that an equation of the second degree in two variables may be factored is given in Chapter

VIII. A method of factoring, *when factoring is possible*, is shown in the following example: [See also Art. 97]

$$2x^2 - xy - 3y^2 + 9x + 4y + 7.$$

Take out 2 as a factor, complete the square of the x-terms, and collect the other terms: the expression takes the form of the difference between two squares:

$$2\left[x^2 + \left(\frac{9-y}{2}\right)x + \left(\frac{9-y}{4}\right)^2 - \frac{25}{16}(y^2 - 2y + 1)\right]$$

$$= 2\left\{\left[x + \frac{9-y}{4}\right]^2 - \left[\frac{5}{4}(y-1)\right]^2\right\}$$

$$= 2\left[x + \frac{9-y}{4} + \frac{5}{4}(y-1)\right]\left[x - \frac{9-y}{4} - \frac{5}{4}y - 1)\right]$$

$$= (x + y + 1)(2x - 3y + 7).$$

EXERCISES

The following equations represent pairs of straight lines. Find in each case the equations of the lines, their common point, and the angle between them:

1. $6y^2 - xy - x^2 + 30y + 36 = 0.$
2. $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0.$
3. ? $x^2 - 2xy \sec \alpha + y^2 = 0.$
4. $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0.$
5. The equations of the opposite sides of a parallelogram are

$$x^2 - 7x + 6 = 0 \text{ and } y^2 - 14y + 40 = 0.$$

Find the equations of the diagonals, and their common point.

? 6. Show that $6x^2 + 5xy - 6y^2 = 0$ is the equation of the bisectors of the angles made by the lines $2x^2 + 12xy + 7y^2 = 0$,

EXAMPLES ON CHAPTER IV

1. The points $(^-2, 4)$ and $(6, ^-4)$ are the extremities of the base of an equilateral triangle. Find the equations of the sides, and the coördinates of the third vertex. (Two solutions).

2. Three of the vertices of a parallelogram are at the points $(2, 2)$, $(6, 8)$, and $(10, ^-4)$. Find the fourth vertex. (Three solutions). Find also the area of the parallelogram.

3. Find the equations of the two lines drawn through the point $(3, 0)$, such that the perpendiculars let fall from the point $(^-6, ^-6)$ upon them are each of length 3.

4. Perpendiculars are let fall from the point $(5, 0)$ upon the sides of the triangle whose vertices are at the points $(4, 3)$, $(^-4, 3)$, and $(0, ^-5)$. Show that the feet of these three perpendiculars lie on a straight line.

Find the equation of the straight line:

5. Through the point $(1, 2)$ and the point of intersection of the lines $x - y = 0$ and $x + y - 2 = 0$. Prove that it is a bisector of the angle formed by the two given lines.

6. Through the intersection of the lines $2x - 3y + 2 = 0$ and $3x - 4y - 2 = 0$ and parallel to $5x - 2y + 3 = 0$.

7. Through the point $(1, 2)$, and intersecting the line $x + y = 4$ at a distance $\frac{1}{3}\sqrt{6}$ from this point.

8. Find the equation of a straight line through the point $(5, 0)$ and making equal angles with the lines $y - 2x + 6 = 0$ and $y = -2x + 16$.

9. Prove analytically that the diagonals of a square are of equal length, bisect each other, and are at right angles.

10. Prove analytically that the line joining the middle points of two sides of a triangle is parallel to the third side and equal to half its length.

11. Find the locus of the vertex of a triangle whose base is $4a$ and the difference of the squares of whose sides is $16c^2$. Trace the locus.

12. Find the equations of the lines from the vertex (4, 3) of the triangle of Ex. 4, trisecting the opposite side. What are the ratios of the areas of the resulting triangles?

13. A point moves so that the sum of its distances from the lines $3y - x + 11 = 0$ and $2x - 7y + 1 = 0$ is 4. Find the equation of its locus. Draw the figure.

14. Find the equation of the path of the moving point of Ex. 13, if the distances from the fixed lines are in the ratio 3 : 4.

15. Find the equation of a straight line through the intersection of $y = 7x - 4$ and $2x + y = 5$, and forming with the x-axis the angle $\frac{\pi}{3}$.

16. Find the equation of the locus of a point which moves so as to be always equidistant from the points (4, 0) and (0, $^-4$).

17. Find the equation of the locus of a point which moves so as to be always equidistant from the points (0, 0) and (6, 4). Show that the points (0, 0), (6, 4), and (2, $^-2$) are the vertices of an isosceles triangle.

18. Find the center and radius of the circle circumscribed about the triangle whose vertices are the points (2, 2), (12, 2), (7, $^-3$).

19. Find analytically the equation of the locus of the vertex of a triangle having its base and area constant.

20. Prove analytically that the locus of a point equidistant from two given points (x_1, y_1) and (x_2, y_2) is the perpendicular bisector of the line joining the given points.

21. The base of a triangle is of length 2, and is given in position; the difference of the squares of the other two sides is 5; find the equation of the locus of its vertex.

22. What lines are represented by the equations:
(α) $x^2y=xy^2$; (β) $14x^2-5xy-y^2=0$; (γ) $xy+bx+ay+ab=0$?

23. What must be the value of m in order that the lines $y-x-1=0$, $y-2x-2=0$, and $y-mx-3=0$ shall pass through a common point?

24. By finding the area of the triangle formed by the three points $(^-4b, 0)$, $(0, 4a)$ and $(^-3b, a)$, prove that these three points are in a straight line. Prove this also by showing that the third point is on the line joining the other two.

25. Find by the method of Art. 36 the point of intersection of the two lines $2x-3y+7=0$ and $4x=6y+2$; and interpret the result by means of Art. 38.

26. Find the equations of two lines each drawn through the point (3, 4), and forming with the axes a triangle whose area is $^-8$.

27. Find the equation of a line through the point $(^-4, 3)$, such that the portion between the axes is divided by the given point in the ratio 5 : 3.

28. Find the equation of the perpendicular erected at the middle point of the line joining (5, 2) to the intersection of the two lines

$$x+2y=11 \quad \text{and} \quad 9x-2y=59.$$

29. A point moves so that the square of its distance from the origin equals twice the square of its distance from the y-axis; find the equation of its locus.

30. Find the equations to the four sides of a square, the coördinates of two of its opposite angular points being (4, 6), (6, 8).

? **31.** Show that the equation

$$56x^2 + 441xy - 56y^2 - 79x - 47y + 9 = 0$$

represents the bisectors of the angles between the straight lines represented by

$$15x^2 - 16xy - 48y^2 - 2x + 16y - 1 = 0.$$

? **32.** Two lines are represented by the equation

$$Px^2 + 2Rxy + Qy^2 = 0.$$

Find the angle between them.

33. Prove that the three altitudes of a triangle meet in a point.

CHAPTER V

THE CIRCLE

SPECIAL EQUATION OF THE SECOND DEGREE

$$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0$$

54. It must be kept clearly in mind that one of the chief aims of an elementary course in Analytic Geometry is to teach a new *method* for the study of geometric properties of curves and surfaces. Power and facility in the use of such a new method are best acquired by applying it first to those loci whose properties are already best understood. Accordingly, the straight line having already been studied in Chapter IV, the circle will next be examined.

55. The circle: its definition, and equation. The circle may be defined as the path traced by a point which moves in a plane in such a way as to be always at a constant distance from a given fixed point. This fixed point is the **center**, and the constant distance is the **radius**, of the circle.

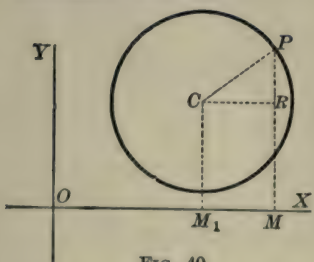

FIG. 40.

To derive the equation from this definition, let $C \equiv (h, k)$ be the center, r the radius, and $P \equiv (x, y)$ any point on the curve. Also draw the ordinates M_1C and MP, and the line CR parallel to the x-axis;

then $CP = r$;

but [Art. 19], $CP = \sqrt{(x-h)^2 + (y-k)^2}$,

$\therefore$ $r = \sqrt{(x-h)^2 + (y-k)^2}$; N. B

i.e., $(x-h)^2 + (y-k)^2 = r^2$, [16]

which is the required equation of the circle.

With given fixed axes, equation [16] may, by rightly choosing h, k, and r, represent any circle whatever; it is, therefore, called the *general equation of the circle.* Of its special forms, the one for which the center coincides with the origin is of very frequent application; in that case $h = k = 0$, and equation [16] becomes

$$x^2 + y^2 = r^2. \qquad [17]$$

N. B.

EXERCISES

First construct the circle, then find its equation, being given:

1. The center (6, ⁻4), the radius 5.
2. The center (0, ⁻3), the radius $\frac{3}{2}$.
3. The center (4, ⁻4), the radius 4.
4. The center (0, 0), the radius 5.
5. The center (0, ⁻4), the radius 3.
6. How are circles related for which h and k are the same, while r is different for each? for which h and r are the same, while k differs for each?
7. What form does the equation of the circle assume when the center is on the x-axis and the origin on the circumference? when the circle touches each axis and has its center in quadrant II?
8. In what respect is equation [16] more general than equation [17]?

56. In rectangular coördinates every equation of the form $Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$ represents a circle. The equations of the circles already obtained (equations [16] and [17], as well as the answers to examples 1 to 5 and 7) are all of the form

$$x^2 + y^2 + 2Gx + 2Fy + C = 0. \tag{1}$$

It will now be shown that, for all values of G, F, and C, for which the locus of equation (1) is real, this equation represents a circle.

To prove this, it is only necessary to complete the square in the x-terms and in the y-terms, and then transpose C to the second member. Equation (1) may then be written in the form

$$\begin{aligned}(x+G)^2 + (y+F)^2 &= G^2 + F^2 - C \\ &= (\sqrt{G^2 + F^2 - C})^2,\end{aligned} \tag{2}$$

which is (cf. equation [16]) the equation of a circle whose center is the point $(-G, -F)$, and whose radius is

$$\sqrt{G^2 + F^2 - C}.$$

It follows that every equation of the more general form

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

represents a circle; for, by Art. 35, this equation has the same locus as has $x^2 + y^2 + 2\frac{G}{A}x + 2\frac{F}{A}y + \frac{C}{A} = 0$, and this last equation is of the form of equation (1), which, as shown above, represents a circle.

Hence, interpreted in rectangular coördinates, every equation of the second degree from which the term in xy is absent, and in which the coefficient of x^2 equals that of y^2, represents a circle.

57. Equation of a circle through three given points. By means of equation [16], or of the equation

$$x^2 + y^2 + 2Gx + 2Fy + C = 0, \tag{1}$$

which has been shown in Art. 56 to be its equivalent, the problem of finding the equation of a circle which shall pass through any three given points not lying on a straight line can be solved; *i.e.*, the constants h, k, and r, or G, F, and C, may be so determined that the circle shall pass through the three given points.

To illustrate: let the given points be (1, 1), (2, ⁻1), and (3, 2), and let $x^2+y^2+2\,Gx+2\,Fy+C=0$ be the equation of the circle that passes through these points; to find the values of the constants G, F, and C.

Since the point (1, 1) is on this circle, therefore [cf. Art. 28],
$$1+1+2\,G+2\,F+C=0;$$
similarly,
$$4+1+4\,G-2\,F+C=0,$$
and
$$9+4+6\,G+4\,F+C=0.$$

These equations give: $G=-\frac{5}{2}$, $F=-\frac{1}{2}$, and $C=4$. Substituting, the equation of the required circle becomes
$$x^2+y^2-5\,x-y+4=0;$$
its center is at the point $(\frac{5}{2}, \frac{1}{2})$, while its radius is $\frac{1}{2}\sqrt{10}$.

NOTE. The algebraic fact that the most general equation of the circle contains three parameters (h, k, and r, or G, F, and C, above) corresponds to the geometric property that a circle is uniquely determined by three of its points.

EXERCISES

Find the radii, and the coördinates of the centers, of the following circles; also, draw the circles.

1. $x^2+y^2-3\,x-4\,y-41=0$.
2. $3\,x^2+3\,y^2-5\,x-7\,y+1=0$.
3. $x^2+y^2=3\,(x+3)$.
4. $2(x^2+y^2)=7\,y$.
5. $ax^2+ay^2=-bx-cy$.
6. $(x+y)^2+(x-y)^2=8\,a^2$.

7. What loci are represented by the equations

$$(x+h)^2+(y+k)^2=0,$$

and $$x^2+y^2-2x+6y+38=0\,?$$

Find the equation of the circle through the points:

8. $(1, 2)$, $(3, {}^-4)$, and $(5, {}^-6)$;

9. $(0, 0)$, (a, b), and (b, a);

10. $(10, 9)$, $(4, {}^-5)$, $(0, 5)$;

11. $({}^-5, {}^-2)$, $({}^-2, 2)$ and having the radius $2\frac{1}{2}$.

12. Find the equation of the circle which has the line joining the points $(1, 2)$ and $(5, {}^-1)$ for a diameter.

13. Find the equation of the circle which touches each axis, and passes through the point $(8, 4)$.

14. A circle has its center on the line $3y+4x=7$, and touches the two lines $x+y=3$ and $y-x=3$; find its equation, radius, and center; also draw the circle.

15. Show that equation (1) represents a real, imaginary, or "point" circle (*i.e.*, only one real point), according as G^2+F^2-C is respectively greater than, less than, or equal to, zero.

SECANTS, TANGENTS, AND NORMALS

58. Definitions of secants, tangents, and normals. A straight line will, in general, intersect any given curve in two or more distinct points; it is then called a **secant** line to the curve. Let P_1 and P_2 be two successive points of intersection of a secant line P_1P_2Q with a given curve $LP_1P_2 \cdots K$; if this secant line be rotated about the point P_1 so that P_2 approaches P_1 along the curve, the limiting position P_1T which the secant

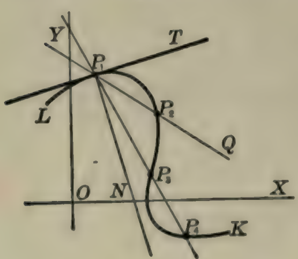

FIG. 41.

approaches, as P_2 approaches coincidence with P_1, is called a **tangent** to the curve at that point. This conception of the tangent leads to a method of extensive application for deriving its equation, — the so-called "secant method."

Since the points of intersection of a line and a curve are found [Art. 36] by considering their equations as simultaneous, and solving for x and y, it follows that, if the line is tangent to the curve, the abscissas of two points of intersection, and also their ordinates, are equal. Therefore, if the line is a tangent, *the equation obtained by eliminating x or y between the equation of the line and that of the curve must have a pair of equal roots.*

A straight line drawn perpendicular to a tangent and through the point of tangency is called a **normal** line to the curve at that point. Thus, in Fig. 41, P_1P_2, P_1P_3 are secants, P_1T is a tangent, and P_1N a normal to the curve at P_1.

59. Tangents: Illustrative examples.

(1) Find the equation of that tangent to the circle $x^2+y^2=5$, which makes an angle of 45° with the x-axis.

Since the line makes an angle of 45° with the x-axis, its equation is $y=x+b$, where b is to be determined so that this line shall *touch* the circle.

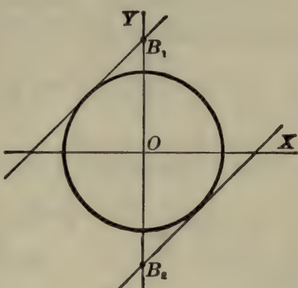

FIG. 42.

Clearly, from the figure, there are two values of b (OB_1 and OB_2) for which this line will be tangent to the circle. According to Art. 58, these values of b are those which make the two points of intersection of the line and the circle become coincident.

Considering the equations $x^2+y^2=5$ and $y=x+b$ as simulta-

neous, and eliminating y, the resulting equation in x is

$$x^2+(x+b)^2=5,$$

i.e., $$2x^2+2bx+b^2-5=0.$$

The roots of this equation will become equal, *i.e.*, the abscissas of the points of intersection will be equal [Art. 6], if

$$b^2-2(b^2-5)=0, \quad i.e., \quad \text{if } b=\pm\sqrt{10}.$$

The equations of the two required tangent lines are, therefore,

$$y=x+\sqrt{10}, \quad \text{and} \quad y=x-\sqrt{10}.$$

(2) Find the equations of those tangents to the circle $x^2+y^2=6y$ that are parallel to the line $x+2y+11=0$.

The equation of a line parallel to $x+2y+11=0$ is $x+2y+k=0$, where k is an arbitrary constant [Art. 49], and this line will become tangent to the circle, if the value of the constant k be so chosen that the two points in which the line meets the circle shall become coincident.

Considering the equations $x^2+y^2=6y$ and $x+2y+k=0$ as simultaneous, and eliminating x, the resulting equation in y is $(-k-2y)^2+y^2=6y$, *i.e.*, $5y^2+(4k-6)y+k^2=0$.

The two values of y will be equal if [Art. 6]

$$(4k-6)^2-20k^2=0, \quad i.e., \quad \text{if } k^2+12k-9=0,$$

i.e., if $$k=-6\pm3\sqrt{5},$$

and the two required tangent lines are:

$$x+2y-6+3\sqrt{5}=0, \quad \text{and} \quad x+2y-6-3\sqrt{5}=0.$$

EXERCISES

Find the equations of the tangents:

1. To the circle $x^2+y^2=4$, parallel to the line $x+2y+a=0$.

2. To the circle $3(x^2+y^2)=4y$, perpendicular to the line $x+y=b$.

3. To the circle $x^2+y^2+10x-6y-2=0$, parallel to the line $y=2x+7$.

4. To the circle $x^2+y^2=r^2$, and forming with the axes a triangle whose area is r^2.

5. Show that the line $y=x+c\sqrt{2}$ is, for all values of c, tangent to circle $x^2+y^2=c^2$; and find, in terms of c, the point of contact.

6. Prove that the circle $x^2+y^2-2x-2y+1=0$ touches both coördinate axes; and find the points of contact.

7. For what values of c will the line $3x+4y+c=0$ touch the circle $x^2+y^2-8x+12y-44=0$?

8. For what value of r will the circle $x^2+y^2=r^2$ touch the line $y=3x+5$?

9. Prove that the line $bx=a(y-a)$ touches the circle $x(x+b)+y(y-a)=0$; and find the point of contact.

10. Three tangents are drawn to the circle $x^2+y^2=25$; one of them is parallel to the x-axis, and together they form an equilateral triangle. Find their equations, and the area of the triangle.

60. Equation of tangent to the circle $x^2+y^2=r^2$ in terms of its slope. The equation of the tangent to a given circle, in terms of its slope, is found in precisely the same way as that followed in solving (1) of Art. 59. Let m be the given slope of the tangent, then the equation of the tangent is of the form

$$y=mx+b; \tag{1}$$

and b is found to be

$$b = \pm r\sqrt{1+m^2}.$$

Therefore, $$y = mx \pm r\sqrt{1+m^2}, \qquad [18]$$

is, for all values of m, tangent to the circle (2).

Equation [18] enables one to write at once the equation of the tangent of given slope, and is also of service in getting the equation of a tangent through an outside point; but only, of course, *to a circle whose center is at the origin.*

E.g., to find the equation of the tangent whose slope $m = \tan 45° = 1$, to the circle $x^2 + y^2 = 5$; substitute 1 for m and $\sqrt{5}$ for r in equation [18]. This gives $y = x \pm \sqrt{10}$.

Again, to find the equation of the tangent to the circle $x^2 + y^2 = 25$, from the outside point $P \equiv (^+7, ^-1)$. Now the coördinates of P satisfy equation [18],

hence, $$y = mx \pm 5\sqrt{1+m^2};$$

i.e., $$^-1 = {}^+7\,m \pm 5\sqrt{m^2+1}.$$

Hence, $12\,m^2 + 7\,m - 12 = 0$ and $m_1 = \frac{3}{4}$, $m_2 = -\frac{4}{3}$;

i.e., there are two such tangent lines, and their equations are

$$y = \tfrac{3}{4}x - \tfrac{25}{4} \text{ and } y = -\tfrac{4}{3}x + \tfrac{25}{3}.$$

EXERCISES

Find the equations of the lines which are tangent:

1. To the circle $x^2 + y^2 = 25$, and whose slope is 4;

2. To the circle $x^2 + y^2 = 4$, and which are parallel to the line $x + 3\,y + 3 = 0$;

3. To the circle $x^2 + y^2 = 9$, and which make an angle of 30° with the x-axis; with the y-axis;

4. To the circle $x^2 + y^2 = r^2$, at the point (x_1, y_1);

5. To the circle $x^2+y^2=25$, and perpendicular to the line joining the points $(^-3, 7)$ and $(7, 5)$.

6. To the circle $x^2+y^2=2x+2y-1$, and whose slope is $^+1$.

7. To the circle $x^2+y^2+2Gx+2Fy+C=0$, with slope m.

61. Equation of tangent to the circle in terms of the coördinates of the point of contact: the secant method.

(*a*) *Center of the circle at the origin.* Let $P_1 \equiv (x_1, y_1)$ be the point of tangency, on the given circle

$$x^2+y^2=r^2. \tag{1}$$

Through P_1 draw a secant line LM, and let $P_2 \equiv (x_2, y_2)$ be its other point of intersection with the circle. If the point P_2 moves *along the circle* until it comes into coincidence with P_1, the limiting position of the secant LM is the tangent P_1T. [Art. 58].

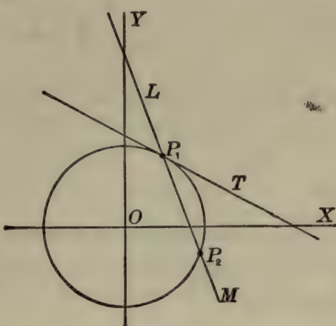

FIG. 43.

The equation of the line LM is of the form

$$y-y_1=m(x-x_1), \tag{2}$$

in which the special value of the slope

$$m=\frac{y_2-y_1}{x_2-x_1} \tag{3}$$

is to be determined by the fact that the fixed point P_1 and the moving point P_2 are always on the circle (1);

therefore $$x_1^2+y_1^2=r^2; \tag{4}$$

and $$x_2^2+y_2^2=r^2. \tag{5}$$

Hence, subtracting equation (4) from equation (5),

$$x_2^2-x_1^2+y_2^2-y_1^2=0;$$

that is, $(y_2 - y_1)(y_2 + y_1) = -(x_2 - x_1)(x_2 + x_1)$;

whence, $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.\text{*} \qquad (6)$$

Substituting this value of m in equation (2) gives

$$y - y_1 = -\frac{x_2 + x_1}{y_2 + y_1}(x - x_1), \qquad (7)$$

which is the equation of the secant line LM of the given circle (1).

Now let P_2 move along the circle until it coincides with P_1, *i.e.*, until $x_2 = x_1$, and $y_2 = y_1$, then equation (6) becomes

$$y - y_1 = -\frac{x_1 + x_1}{y_1 + y_1}(x - x_1);$$

i.e., $$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

which, by clearing of fractions and transposing, may be written in the form $$x_1x + y_1y = x_1^2 + y_1^2;$$

i.e., $$x_1x + y_1y = r^2, \qquad \text{N.B} \qquad [19]$$

which is the required equation of the tangent to the circle $x^2 + y^2 = r^2$, x_1 and y_1 being the coördinates of the point of tangency.

NOTE. If P_2 approaches P_1 so that finally $x_2 = x_1$, $y_2 = y_1$, in equation (2), then the indeterminate form $y - y_1 = \frac{0}{0}(x - x_1)$ is obtained. This indeterminateness arises because account has not been taken of the condition that P_1 and P_2 are on the circle, and therefore there is no definiteness to the *path* by which P_2 approaches P_1.

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* The difference between equations (3) and (6) consists in this: no matter where the points (x_1, y_1) and (x_2, y_2) may be, equation (3) represents the slope of the straight line passing through them; but equation (6) gives the slope of the line through these points only when they are on the circle $x^2 + y^2 = r^2$.

(*b*) *Center of circle not at origin.* By a similar process, if the equation of the given circle is

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0, \tag{8}$$

then the equation of a secant line through the two points P_1 and P_2 *on the circle* will be found to be

$$y - y_1 = -\frac{x_2 + x_1 + 2\,G}{y_2 + y_1 + 2\,F}(x - x_1);^* \tag{9}$$

which for the tangent at the point P_1 becomes

$$y - y_1 = -\frac{x_1 + G}{y_1 + F}(x - x_1);^* \tag{10}$$

and this may be reduced to the form

$$\boldsymbol{x_1x + y_1y + G(x + x_1) + F(y + y_1) + C = 0,} \qquad [20]$$

which is the required equation of the tangent to the circle (8), x_1 and y_1 being the coördinates of the point of contact.*

Note. Equation [20] may be easily remembered if it is observed that it differs from the equation of the circle (equation [7]) only in having x_1x, y_1y, $x + x_1$, and $y + y_1$ in place of x^2, y^2, $2\,x$, and $2\,y$, respectively. It will be found later that any equation of the second degree (from which the xy-term is absent) bears this same relation to the equation of a tangent to its locus, x_1 and y_1 being the coördinates of the point of contact. Compare, also, equation [19] with equation (1).

It must be carefully kept in mind that equations [19] and [20] represent tangents *only if* (x_1, y_1) *is a point on the circle.* It may be shown that these equations represent other lines if (x_1, y_1) is *not* on the circle, called *polars* of the given point.

62. Equation of a normal to a given circle. By definition [Art. 58] the normal at a given point, $P_1 \equiv (x_1, y_1)$, on any curve is the line through P_1, and perpendicular to the tangent at P_1. Hence, to get the equation of the normal at any given point, it

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* Students should carefully work out all the details here.

is only necessary to write the equation of the tangent at this point [Art. 61], and then the equation of a line perpendicular to this tangent [Arts. 43, 49] and passing through the given point. Thus the equation of the normal to the circle

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

at the point $P_1 \equiv (x_1, y_1)$, is

$$y - y_1 = \frac{y_1 + F}{x_1 + G}(x - x_1). \tag{2}$$

Since the coördinates $-G$ and $-F$ of the center of the given circle (1) satisfy equation (2), *hence, every normal to a circle passes through the center of the circle.*

If the center of the circle is at the origin, then $G = 0$, $F = 0$, and $C = -r^2$, and the equation (2) of the normal becomes

$$y - y_1 = \frac{y_1}{x_1}(x - x_1), \tag{3}$$

which reduces to $x_1 y - x y_1 = 0$, an equation which could have been derived for the circle $x^2 + y^2 = r^2$ in precisely the same way that equation (2) was derived from equation (1).

EXERCISES

1. Derive, by the secant method, the equation of the tangent to the circle $x^2 + y^2 - 2\,rx = 0$, the point of contact being $P_1 \equiv (x_1, y_1)$.

2. Write the equation of the tangent to the circle:

(α) $x^2 + y^2 = 41$, the point of contact being (4, 5).

(β) $x^2 + y^2 - 3\,x + 10\,y = 15$, the point of contact being $(4, {}^{-}11)$.

(γ) $(x + 2)^2 + (y + 3)^2 = 10$, the point of contact being $({}^{-}5, {}^{-}4)$.

(δ) $3\,x^2 + 3\,y^2 + 2\,y + 4\,x = 0$, the point of contact being (0, 0).

3. Find the equation of the normal to each of the circles of Ex. 2, through the given point.

4. A tangent to a circle is perpendicular to the radius drawn to its point of contact. By means of this fact, derive the equation of the tangent to the circle $(x-a)^2+(y-b)^2=r^2$ at the point (x_1, y_1). (Cf. equation [20]).

5. From the fact that a normal to a circle passes through its center, find the equation of the normal to the circle $x^2+y^2-8x+6y+21=0$ at the point $(4, {}^-1)$.

6. Find the equations of the two tangents, drawn through the external point (11, 3) to the circle $x^2+y^2=40$.

7. What is the equation of the circle whose center is at the point (5, 3), and which touches the line $3x+2y+10=0$?

8. If the line $\frac{x}{m}+\frac{y}{n}=1$ touches the circle $x^2+y^2=r^2$, find the equation connecting m, n, and r.

9. Find the equation of a circle inscribed in a triangle whose sides are the lines $x=0$, $y=0$, and $\frac{x}{a}+\frac{y}{b}=1$. ($a$ and b being positive).

10. Solve Ex. 6 by assuming x_1 and y_1 as the coördinates of the point of contact, and then finding their numerical values from the two equations which they satisfy.

11. Prove that from an outside point two tangents and only two, can be drawn to a circle.

63. Lengths of tangents and normals. Subtangents and subnormals. The tangent and normal lines of any curve extend indefinitely in both directions; it is, however, convenient to consider as the **length of the tangent** the length TP_1, measured from the point of intersection (T) of the tangent with the x-axis to the point of tangency (P_1); and simi-

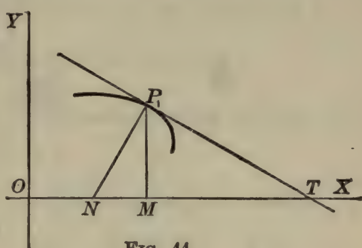

FIG. 44.

larly to consider as the **length of the normal** the length NP_1, measured from the point of intersection (N) of the normal with the x-axis to P_1.

The **subtangent** is the length TM, where M is the foot of the ordinate of the point of tangency P_1; and the **subnormal** is the corresponding length NM. As thus taken, the subtangent and the subnormal are of the opposite sign; ordinarily, however, one is concerned merely with their *absolute* values, irrespective of the algebraic sign. The subtangent is the projection of the tangent length on the x-axis, and the subnormal is the like projection of the normal length.

64. Tangent and normal lengths, subtangent and subnormal, for the circle. The definitions given in the preceding article furnish a direct method for finding the tangent and normal lengths, as well as the subtangent and subnormal, for a circle.

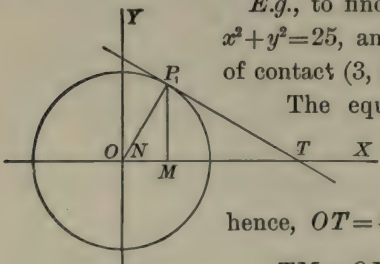

Fig. 45.

E.g., to find these values for the circle $x^2+y^2=25$, and corresponding to the point of contact (3, 4), proceed thus:

The equation of the tangent P_1T is [Art. 61]

$$3x+4y=25;$$

hence, $OT=\frac{25}{3}$ and the subtangent is

$$TM=OM-OT=3-\tfrac{25}{3}=-5\tfrac{1}{3}.$$

Similarly, the tangent length is

$$TP_1=\sqrt{\overline{MT}^2+\overline{MP_1}^2}=\sqrt{(\tfrac{16}{3})^2+4^2}=6\tfrac{2}{3}.$$

Again, the equation of the normal at (3, 4) is

$$4x-3y=0;$$

hence, the subnormal is $\quad OM=+3,$

and the normal is $\quad OP_1=5.$

EXERCISES

Find the lengths of the tangent, subtangent, normal, and subnormal:

1. For the point $(^-4, ^+11)$ on the circle $x^2+y^2+3x-10y=15$;

2. For the point $(^-1, ^+3)$ on the circle $x^2+y^2+10x=0$.

3. For the point whose abscissa is 3 on the circle $x^2+y^2=25$.

4. The subtangent for a certain point on a circle, whose center is at the origin, is $5\frac{1}{2}$, and its subnormal is 3. Find the equation of the circle, and the point of tangency.

5. Show that $TM=\frac{y_1}{m}$, and $NM=-y_1m$, where m is the slope of the tangent.

6. Show that the square of the length of the tangent from the outside point (x_1, y_1) to the circle $x^2+y^2=r^2$ is $x_1^2+y_1^2-r^2$.

65. Circles through the intersections of two given circles. Given two circles whose equations are

$$x^2+y^2+2G_1x+2F_1y+C_1=0, \tag{1}$$

and

$$x^2+y^2+2G_2x+2F_2y+C_2=0. \tag{2}$$

These circles intersect, in general, in two finite points $P_1\equiv(x_1, y_1)$ and $P_2\equiv(x_2, y_2)$, and [Art. 38] the equation

$$x^2+y^2+2G_1x+2F_1y+C_1 \\ +k(x^2+y^2+2G_2x+2F_2y+C_2)=0, \tag{3}$$

where k is any constant, represents a curve which passes through these same common points.

The locus of equation (3) is, moreover, a circle [Art. 56]; hence, a series of different values being assigned to the parameter k, equation (3) represents what is called a "family" of

circles; each one of these circles passing through the two points P_1 and P_2 in which the given circles (1) and (2) intersect each other.

66. Common chord of two circles. If in equation (3), Art. 65, the parameter k be given the particular value -1, the equation reduces to

$$2(G_1 - G_2)x + 2(F_1 - F_2)y + C_1 - C_2 = 0, \tag{4}$$

which is of the first degree, and therefore represents a straight line; but this locus belongs to the family represented by equation (3) of Art. 65, hence it passes through the two points P_1 and P_2 in which the circles (1) and (2) intersect. This line (4) is, therefore, the **common chord** of these circles.

67. Radical axis; radical center. The angle formed by two intersecting curves. The line whose equation is obtained by eliminating the x^2 and y^2 terms between the equations of two given circles, as in Art. 66, whether the circles intersect in real points or not, is called the **radical axis** of the two circles. If the two given circles intersect each other in real points, then this line is called also their common chord; that is, the common chord of two circles is a special case of the radical axis of two circles.

Three circles, taken two and two, have three radical axes. It is easily shown that these three radical axes pass through a common point; this point is called the **radical center** of the three circles.

By the angle between two intersecting curves is meant the angle formed by the two tangents, one to each curve, drawn through the point of intersection.

Hence to find the angle at which two curves intersect, it is necessary to find the point of intersection, then to find the equations of the tangents at this point, one to each curve, and finally to find the angle formed by these tangents.

EXERCISES

1. Find the equation of the common chord of the circles

$$x^2+y^2-8y-4x-14=0, \quad x^2+y^2+8y-8x-2=0.$$

2. Find the points of intersection of the circles in exercise 1, and the length of their common chord.

3. Find the radical axis, and also the length of the common chord, for the circles $x^2+y^2+ax+by+c=0$, $x^2+y^2+bx+ay+c=0$.

4. Find the radical center of the three circles

$$x^2+y^2+4x+7=0,$$
$$2(x^2+y^2)+3x+5y+9=0,$$
$$x^2+y^2+y=0.$$

5. Show that tangents from the radical center, in exercise 4, to the three circles, respectively, are equal in length.

6. Prove analytically that the tangents to two circles from any point on their radical axis are equal.

7. Find the angle at which the circle $x^2+y^2=25$ cuts the circle $(x-7)^2+y^2+2y-24=0$.

8. At what angle does the circle $(x-4)^2+y^2-2y=15$ meet the line $x+2y=4$?

9. Show that if in Art. 65, equation (3), the radius becomes infinite, the common chord is obtained, as in Art. 66, equation (4).

10. For the circles of Ex. 1, show that the radical axis is perpendicular to the line of centers.

11. The radical axis is perpendicular to the line joining the centers of two circles.

EXAMPLES ON CHAPTER V

1. Find the equation of the circle circumscribing the triangle whose vertices are at the points $(10, 9)$, $(4, {}^-5)$, and $(0, 5)$. What is its center? its radius?

2. Determine the center of the circle

$$(x-a)^2+(y-b)^2=a^2+b^2.$$

What family of circles is represented by this equation, if a and b vary under the one restriction that a^2+b^2 is to remain constant?

3. What must be the relations among the coefficients in order that the circles

$$x^2+y^2-2\,G_1x-2\,F_1y+C_1=0,$$

and

$$x^2+y^2-2\,G_2x-2\,F_2y+C_2=0,$$

shall be concentric? that they shall have equal areas?

4. Under what limitations upon the coefficients is the circle

$$x^2+y^2+Dx+Ey+F=0$$

tangent to each of the axes?

5. Find the equation of the circle which has its center on the x-axis, and which passes through the origin and also through the point $({}^-2, {}^-3)$.

6. Find the points of intersection of the two circles

$$x^2+y^2-4\,x-2\,y-31=0 \text{ and } x^2+y^2-4\,x+2\,y+1=0.$$

7. Circles are drawn having their centers at the vertices of the triangle $(7, 2)$, $({}^-1, {}^-4)$, and $(3, 3)$, respectively, and each passing through the center of a fourth circle which circumscribes this triangle; find their equations, their common chords, and their radical center.

8. Circles having the sides of the triangle $(7, 2)$, $({}^-1,\ 4)$, $(3, 3)$ as diameters are drawn; find their equations, their radical axes, and their radical center.

9. Find the equation of the circle passing through the origin and the point (x_1, y_1), and having its center on the y-axis.

10. The point $(^-1, ^-1)$ bisects a chord of the circle $x^2 + y^2 = 25$; find the equation of that chord.

11. A circle touches the line $4y + 3x + 3 = 0$ at the point $(3, ^-3)$ and passes through the point $(5, 9)$; find its equation.

12. A circle whose center coincides with the origin touches the line $y - 2x + 3 = 0$; find its equation.

13. At the points in which the circle $x^2 + y^2 + ax + by = 0$ cuts the axes, tangents are drawn; find the equations of these tangents.

14. A circle, whose radius is $\sqrt{74}$, touches the line $5x = 7y - 1$ at the point $(11, 8)$; find the equation of this circle.

15. A circle is inscribed in the triangle $(0, 0)$, $(^-8a, 0)$, $(0, 6a)$; find its equation.

16. Through a fixed point (x_1, y_1) a secant line is drawn to the circle $x^2 + y^2 = r^2$; find the locus of the middle point of the chord which the circle cuts from this secant line, as the secant revolves about the given fixed point (x_1, y_1).

17. Prove analytically that an angle inscribed in a semicircle is a right angle.

18. Prove analytically that a radius drawn perpendicular to a chord of a circle bisects that chord.

19. Angles in the same segment of a circle are equal.

20. Two straight lines touch the circle $x^2 + y^2 - 5x - 3y + 6 = 0$, one at the point $(1, 1)$ and the other at the point $(2, 3)$; find the point of intersection (h, k) of these tangents.

21. Find the condition among the coefficients that must be satisfied if the circles

$$x^2+y^2+2\,G_1x+2\,F_1y=0 \text{ and } x^2+y^2+2\,G_2x+2\,F_2y=0$$

shall touch each other at the origin.

22. Determine F and C so that the circle

$$x^2+y^2+20\,x+2\,Fy+C=0$$

shall cut each of the circles

$$x^2+y^2-4\,x-2\,y+4=0 \text{ and } x^2+y^2+4\,x+2\,y=1$$

at right angles. [Cf. Art. 67].

23. Given the two circles

$$x^2+y^2-4\,x-2\,y+4=0 \text{ and } x^2+y^2+4\,x+2\,y-4=0;$$

find the equation of their common tangents.

24. Find the radical axis of the circles in example 23; show that it is perpendicular to the line joining the centers of the given circles, and find the ratio of the lengths of the segments into which the radical axis divides the line joining the centers.

25. Given the three circles:

$$x^2+y^2-16\,x+60=0, \qquad 3\,x^2+3\,y^2-36\,x+81=0,$$

and

$$x^2+y^2-16\,x-12\,y+84=0;$$

find the point from which tangents drawn to these three circles are of equal length, also find that length. How is this point related in position to the radical center of the given circles? Prove that this relation is the same for any three circles.

26. Find the locus of a point which moves so that the length of the tangent, drawn from it to a fixed circle, is in a constant ratio to the distance of the moving point from a given fixed point.

27. Find the length of the common chord of the two circles

$$(x-a)^2+(y-b)^2=r^2 \quad\text{and}\quad (x-b)^2+(y-a)^2=r^2.$$

From this find the condition that these circles shall touch each other.

28. For what point on the circle $x^2+y^2=9$ are the subtangent and subnormal of equal length? the tangent and normal? the tangent and subtangent?

29. An equilateral triangle is inscribed in the circle $x^2+y^2=25$ with its base parallel to the y-axis; through its vertices tangents to the circle are drawn, thus forming a circumscribed triangle; find the equations, and the lengths, of the sides of each triangle.

30. Find the equation of a circle through the intersection of the circles $x^2+y^2-5=0$, $x^2+y^2-2x-4y+5=0$, and tangent to the line $x+y-3=0$.

31. Find the locus of the vertex of a triangle having given the base $=4a$, and the sum of the squares of its sides $=4c^2$.

32. Find the locus of the middle points of chords drawn through the positive end of the horizontal diameter of the circle $x^2+y^2=a^2$.

33. Through the external point $P_1\equiv(x_1, y_1)$, a line is drawn meeting the circle $x^2+y^2=a^2$ in Q and R; find the locus of middle point of P_1Q as this line revolves about P_1.

34. A point moves so that its distance from the point (1, 3) is to its distance from the point ($^-4$, 1) in the ratio 2 : 3. Find the equation of its locus.

35. Do the circles

$$x^2+y^2-2x-4y-20=0 \text{ and } x^2+y^2-14x-16y+100=0$$

intersect? Show in two ways.

36. Find the equation of a circle of radius $\sqrt{85}$ which passes through the points (1, 2) and (4, $^-3$).

37. Find the equations of the tangents through (2, 3) to the circle $9(x^2+y^2)+6x-12y+4=0$

38. What are the equations of the tangent and the normal to the circle $x^2 + y^2 = 13$, these lines passing through the point $(3, 2)$? through the point $(6, 0)$?

39. At what angle do the circles $x^2 + y^2 - 6y + 2x + 5 = 0$ and $x^2 + y^2 + 4y + 2x - 5 = 0$ intersect each other?

40. A diameter of the circle $4x^2 + 4y^2 + 8x - 12y + 1 = 0$ passes through the point $(1, {}^-1)$. Find its equation, and the equation of the chords which it bisects.

41. Find the locus of a point such that tangents from it to two concentric circles are inversely proportional to the radii of the circles.

42. A and B are two fixed points, and P a point such that $AP = mBP$, where m is a constant; show that the locus of P is a circle, except when $m = 1$.

43. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. Show that the locus is a circle.

44. If m is arbitrary, the circles represented by the equation $(m + 1)(x^2 + y^2) = x + my$ have a common chord.

CHAPTER VI

POLAR COÖRDINATES

68. Of the many kinds of coördinates that may be used to fix the position of a point [Art. 14] thus far only the Cartesian has been studied, and especially the rectangular form, because of its greater symmetry [Art. 17]. But another kind, called polar coördinates, has special advantages for many problems, and is often used. It is implicitly used in the statement 'town A is 50 miles northeast from B'; *i.e.*, the polar coördinates of a point are its direction and distance from a fixed point.

69. Polar coördinates. Given a fixed point O in a fixed directed line OR, and any point P in the plane, with the distance $OP = \rho$ and the angle $ROP = \theta$; then the position of P in the plane is fully determined by ρ and θ.

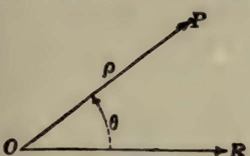

FIG. 46 *a*.

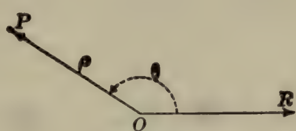

FIG. 46 *b*.

The fixed line OR is called the **initial line** or **polar axis**, the fixed point O the **pole** of the system, and the **polar coördinates**

of the point P are the **radius vector** ρ and the **directional** or **vectorial angle** θ. The usual rule of signs applies to the vectorial angle θ, and the radius vector is positive if measured from O along the terminal side of the angle θ. The point P is designated by the symbol (ρ, θ).

It is clear that *one* pair of polar coördinates (*i.e.*, one value of ρ and one of θ) serve to determine one, and but one, point of the plane. On the other hand, if θ is restricted to values lying between 0 and 2π, then any given point may be designated by *four* different pairs of coördinates. In this respect polar coördinates differ from Cartesian coördinates.

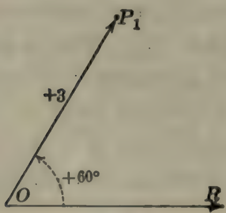

FIG. 47 *a*.

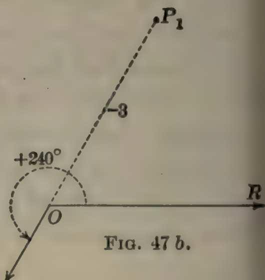

FIG. 47 *b*.

E.g., the polar coördinates (3, 60°) determine one point only, P_1; but P_1 may also be given by the three other pairs of coördinates: ($^-3$, 240°), (3, $^-300$°), and ($^-3$, $^-120$°).

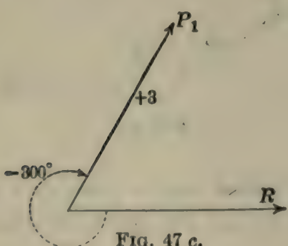

FIG. 47 *c*.

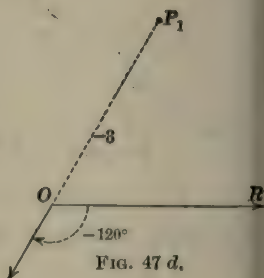

FIG. 47 *d*.

EXERCISES

1. Plot accurately the following points: (2, 20°), $\left(2, \frac{\pi}{9}\right)$, $\left(-7, \frac{\pi}{2}\right)$, $\left(4\pi, \frac{\pi^{(r)}}{3}\right)$, (2, 14 π°), (⁻1, ⁻180°), (7, ⁻45°) (⁻7, 135°), $\left(5, \frac{3\pi}{4}\right)$, $\left(0, \frac{\pi}{3}\right)$, $\left(0, \frac{-\pi}{3}\right)$, (6, 0°), and (⁻6, 0°).

2. Construct the triangle whose vertices are: $\left(4, \frac{\pi}{8}\right)$, $\left(6, \frac{3\pi}{4}\right)$, $\left(2, \frac{5\pi}{4}\right)$; find by measurement the lengths of the sides and the coördinates of their middle points.

3. The base of an equilateral triangle, whose side is 5 inches, is taken as the polar axis, with the vertex as pole; find the coördinates of the other two vertices.

4. Write three other pairs of coördinates for each of the points $\left(2, \frac{\pi}{4}\right)$; (⁻3, 75°); (5, 0°); (0, 60°).

5. Where is the point whose radius vector is 7? whose radius vector is ⁻7? whose vectorial angle is 25°? whose vectorial angle is $0^{(r)}$? whose vectorial angle is ⁻180°?

6. Express each of the conditions of Ex. 5 by means of an equation.

7. What is the direction of the line through the points $\left(a, \frac{\pi}{4}\right)$ and $\left(a, \frac{3\pi}{4}\right)$?

70. Applications. The elementary problems given in Chapters II, III, and IV may often be solved to advantage by the use of polar coördinates. Some elementary formulas are obtained below.

(*a*) *Distance between two points.*

Let OR be the initial line,* O the pole, and let $P_1 \equiv (\rho_1, \theta_1)$ and $P_2 \equiv (\rho_2, \theta_2)$ be the two given fixed points. To find the dis-

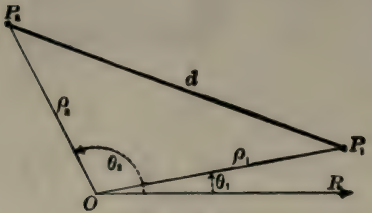

FIG. 48 *a*.

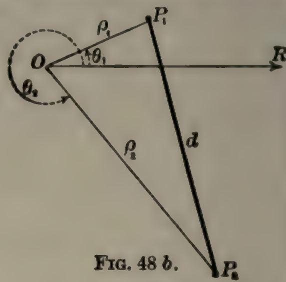

FIG. 48 *b*.

tance $P_1P_2 = d$ in terms of the given constants ρ_1, ρ_2, θ_1, and θ_2.

In the triangle OP_1P_2 [cf. Art. 12]

$$\overline{P_1P_2}^2 = \overline{OP_1}^2 + \overline{OP_2}^2 - 2 \cdot OP_1 \cdot OP_2 \cdot \cos P_1OP_2,$$

i.e., $$d^2 = \rho_1{}^2 + \rho_2{}^2 - 2\,\rho_1\rho_2 \cos(\theta_2 - \theta_1),$$

hence $$d = \sqrt{\rho_1{}^2 + \rho_2{}^2 - 2\,\rho_1\rho_2 \cos(\theta_2 - \theta_1)} \qquad [21]$$

(β) *Area of a triangle.* Let the vertices of the triangle be $P_1 \equiv (\rho_1, \theta_1)$, $P_2 \equiv (\rho_2, \theta_2)$, and $P_3 \equiv (\rho_3, \theta_3)$; to find its area Δ in terms of $\rho_1, \rho_2, \rho_3, \theta_1, \theta_2$, and θ_3.

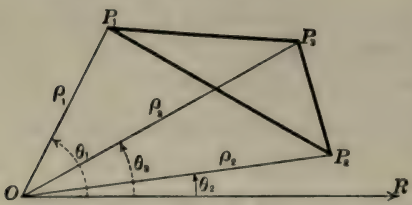

FIG. 49.

Now, $\Delta = OP_2P_3 + OP_3P_1 + OP_1P_2$; †

but $OP_2P_3 = \frac{1}{2}\rho_2\rho_3 \sin(\theta_3 - \theta_2)$, $OP_3P_1 = \frac{1}{2}\rho_3\rho_1 \sin(\theta_1 - \theta_3)$,

and $OP_1P_2 = \frac{1}{2}\rho_1\rho_2 \sin(\theta_2 - \theta_1)$.

$$\therefore\ \Delta = \tfrac{1}{2}\{\rho_1\rho_2 \sin(\theta_2 - \theta_1) + \rho_2\rho_3 \sin(\theta_3 - \theta_2) + \rho_3\rho_1 \sin(\theta_1 - \theta_3)\}. \qquad [22]$$

* The demonstration applies to each figure.

† Area OP_2P_3 and OP_3P_1 are positive; but OP_1P_2 is negative, for the included angle is negative, — from OP_1 to OP_2.

(γ) *Loci.* The study of loci by polar coördinates may be seen from an example:

Given the equation $\rho = 4 \cos \theta$, to find its locus.

This equation is satisfied by the following pairs of values, found as in Art. 27 (3):

$$\begin{array}{ll}
\theta_1 = 0 & \rho_1 = 4 \\
\theta_2 = 30^\circ & \rho_2 = 2\sqrt{3} = 3.46^+ \\
\theta_3 = 60^\circ & \rho_3 = 2 \\
\theta_4 = 45^\circ & \rho_4 = 2\sqrt{2} = 2.8^+ \\
\theta_5 = 90^\circ & \rho_5 = 0 \\
\theta_6 = -30^\circ & \rho_6 = 3.46^+ \\
\theta_7 = -60^\circ & \rho_7 = 2 \\
\theta_8 = -45^\circ & \rho_8 = 2.8^+ \\
\theta_9 = -90^\circ & \rho_9 = 0 \\
\text{etc.} & \text{etc.}
\end{array}$$

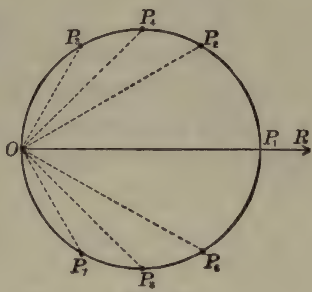

FIG. 50.

The corresponding points are:

$P_1 \equiv (4, 0^\circ)$; $P_2 \equiv (3.46^+, 30^\circ)$; $P_3 \equiv (2, 60^\circ)$; $P_4 \equiv (2.8^+, 45^\circ)$; $P_5 \equiv P_9 \equiv$ the pole $O \equiv (0, \pm 90^\circ)$; $P_6 \equiv (3.46^+, -30^\circ)$; $P_7 \equiv (2, -60^\circ)$; etc.

All these points are found to lie on a circle whose radius is 2, the pole being on the circumference, and the polar axis being a diameter. This circle is the locus of the equation $\rho = 4 \cos \theta$. [Cf. Art. 71 (β), equation 3].

EXERCISES

1. Find the distances between the points $(2, 30^\circ)$, $\left(3, \frac{3\pi}{4}\right)$, and $\left(1, \frac{5\pi}{4}\right)$, taken in pairs.

2. Prove that the points $\left(a, \frac{\pi}{3}\right)$, $\left(a, \frac{2\pi}{3}\right)$, and $(0, 0)$ form an equilateral triangle.

3. Do the points $(7, 30°)$, $(0, 0°)$, and $(^-11, 210°)$ lie on one straight line? Solve this by showing that the area of the triangle is zero, and then verify by plotting the figure.

4. Find the area of the triangle $\left(\pi, \frac{\pi}{2}^{(r)}\right)$, $\left(2\pi, \frac{\pi}{2}^{(r)}\right)$, and $\left(-\pi, \frac{2\pi}{3}^{(r)}\right)$.

5. Find the area and also the perimeter of the triangle whose vertices are the points $(3, 60°)$, $(5, 120°)$, and $(8, 30°)$.

Find by the method of Art. 36 where the following loci cut the polar axis (or initial line):

6. $\rho = 4\sin^2\theta$.

7. $\rho^2 = a^2\cos 2\theta$.

8. A point moves so that its distance from the pole is numerically equal to the tangent of the angle which the straight line joining it to the origin makes with the initial line. Find the polar equation of its locus and plot the figure.

Find the points of intersection of the curves,

9. $\begin{cases} \rho = 9\cos\theta, \\ \rho\cos\theta = 4. \end{cases}$

10. $\begin{cases} \rho = 9\cos(45° - \theta), \\ \rho\cos\left(\frac{\pi}{2} - \theta\right) = 1. \end{cases}$

Discuss and construct the loci of the equations:

11. $\rho^2 = a^2\cos 2\theta$.

12. $\rho = 3\theta$.

13. $\rho = a\sin 2\theta$.

14. Find the equation of a curve passing through the points of intersection of the curves $\rho = 2\cos\theta$, $\rho\cos\theta = \frac{1}{4}$. [Cf. Art. 38].

71. Standard equations of the straight line and circle.

(*α*) *The straight line.*

(1) *Line through two given points.* If point P be *any* point on the straight line through P_1 and P_2, then the area of the triangle PP_1P_2 is zero: hence, (equation [22]), for *every* point of the line P_1P_2 the equation is [Fig. 51]

$$\rho\rho_1 \sin(\theta - \theta_1) + \rho_1\rho_2 \sin(\theta_1 - \theta_2) + \rho_2\rho \sin(\theta_2 - \theta) = 0. \quad [23]$$

(2) *Equation of the line in terms of the perpendicular upon it from the pole, and the angle which this perpendicular makes with the initial line.*

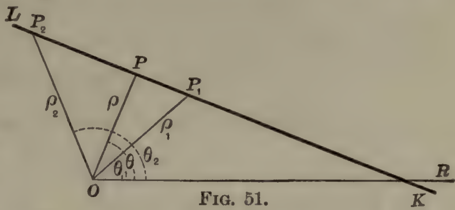

FIG. 51.

Let OR be the initial line, O the pole, and LK the line whose equation is sought. Also, let $N \equiv (p, \alpha)$ be the foot of the perpendicular from O upon LK, and let $P \equiv (\rho, \theta)$ be any other point on LK. Draw ON and OP; then

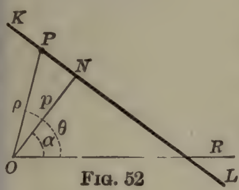

FIG. 52

$$\frac{ON}{OP} = \cos NOP,$$

i.e.,
$$\boldsymbol{\rho \cos(\theta - \alpha) = p,} \quad [24]$$

which is the required equation.

(β) *The circle.* Let OR be the initial line, O the pole, $C \equiv (\rho_1, \theta_1)$ the center of the circle, r its radius, and $P \equiv (\rho, \theta)$ any point on the circle. Draw OC, OP, and CP; then, by trigonometry,

$$r^2 = \rho^2 + \rho_1^2 - 2\,\rho\rho_1 \cos(\theta - \theta_1),$$

i.e., $$\rho^2 - 2\,\rho_1\rho \cos(\theta - \theta_1) + \rho_1^2 - r^2 = 0, \quad [25]$$

which is the equation of the given circle.

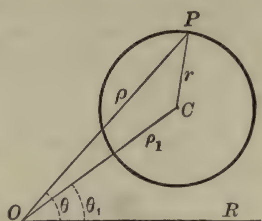

FIG. 53.

By changing the relative positions of the polar axis, the pole, and the center of the circle, equation [25] takes several special forms, which follow. These the student should verify, and plot.

(1) If the center is on the polar axis;

$$\rho^2 - 2\rho_1\rho\cos\theta + \rho_1^2 - r^2 = 0;$$

(2) If the pole is on the circle:

$$\rho - 2r\cos(\theta - \theta_1) = 0;$$

(3) If the pole is on the circle and the polar axis a diameter;

$$\rho - 2r\cos\theta = 0; \qquad [25\,a]$$

(4) If the center is at the pole;

$$\rho = r. \qquad [25\,b]$$

EXERCISES

1. Construct the lines:

(*a*) $\rho\cos(\theta + 30°) = 5$; (*c*) $\rho\cos\left(\theta + \frac{\pi}{4}\right) = 9$;

(*b*) $\rho\sin\theta = 4$; (*d*) $\rho\cos(\theta - \pi) = 10$.

2. Find the polar equations of straight lines at a distance 3 from the pole, and: (1) parallel to the initial line; (2) perpendicular to the initial line.

3. A straight line passes through the points $(10, 60°)$ and $\left(\frac{10\sqrt{3}}{3}, -30°\right)$; find its polar equation.

4. Find the polar equation of a line passing through a given point (ρ_1, θ_1) and cutting the initial line at a given angle $\phi = \tan^{-1} k$.

5. Find the polar coördinates of the point of intersection of the lines

$$\rho\cos\left(\theta - \frac{\pi}{2}\right) = 10, \qquad \rho\cos\left(\theta - \frac{\pi}{6}\right) = 5.$$

6. Derive equation (β) (3) directly from a diagram.

7. Derive equation β (2) directly from a diagram.

8. Construct the circle whose equation is $\rho = 4\cos\theta$.

9. Find the polar equation of the circle whose center is at the point $\left(4, \frac{\pi}{3}\right)$ and whose radius is 5; determine also the points of its intersection with the initial line.

10. Find the polar equation of a circle whose center is at the point $\left(10, \frac{\pi}{2}\right)$ and whose radius is 2. Find also the equations of the tangents to the circle from the pole.

11. A circle of radius 4 is tangent to the two radii vectores which make the angles 60° and 120° with the initial line: find its polar equation, and the distance of the center from the origin.

12. Find the points of intersection of the loci:

(α) $\rho \cos\left(\theta - \frac{\pi}{3}\right) = a$ and $\rho \cos\left(\theta - \frac{\pi}{9}\right) = a$;

(β) $\rho \cos\left(\theta - \frac{\pi}{2}\right) = \frac{3a}{4}$ and $\rho = a \sin\theta$.

Construct the curves:

13. $\rho = 2\theta$.

14. $2\rho = \theta$.

15. $\rho = \sec^2\theta$.

16. $\rho = \sin 2\theta$.

17. $\rho = \frac{1}{1 - \cos\theta}$.

CHAPTER VII

TRANSFORMATION OF COÖRDINATES

72. That the coördinates of a point which remains fixed in a plane are changed by changing the axes to which this fixed point is referred, is an immediate consequence of the definition of coördinates.

It is evident also that the different *kinds* of coördinates of any given point (Cartesian and polar, for example) are connected by definite relations if the elements of reference (the axes) are related in position.

E.g., the point Q, when referred to the polar axis OX and the pole O, has the coördinates $(5, 30°)$, but when it is referred to the rectangular axes OX and OY the coördinates of this same point are $(\frac{5}{2}\sqrt{3}, \frac{5}{2})$.

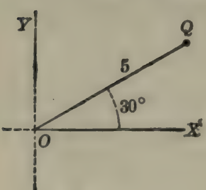

FIG. 54.

Again: while a *curve* remains fixed in a plane, its *equation* may often be greatly simplified by a judicious change of the axes to which it is referred.

E.g., the line L_1L, Fig. 55, when referred to the axes OX and OY, has the equation

$$y = \tan\theta \cdot x + b,$$

but when referred to the axes $O'X'$ and $O'Y'$, the former of which is parallel to the given line, its equation is

$$y = c.$$

For these and other reasons, in the study of curves and surfaces by the methods of analytic geometry, it will often be found advantageous to transform the equations from one set of axes to another.

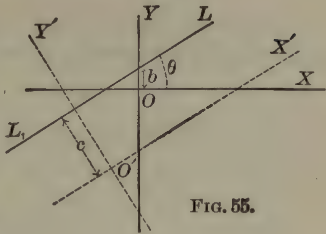

FIG. 55.

Formulas for the simpler changes of axes are derived in the next few articles.*

I. CARTESIAN COÖRDINATES ONLY

73. Change of origin, new axes parallel respectively to the original axes. Let OX and OY be the original axes, $O'X'$ and $O'Y'$ the new axes, respectively parallel to the old, and let the coördinates of the new origin when referred to the original axes be h and k, *i.e.*, $O' \equiv (h, k)$, where $h = OA$ and $k = AO'$. Also let P, any point of the plane, be (x, y) when referred to the first axes OX and OY, and (x', y') when referred to $O'X'$ and $O'Y'$.

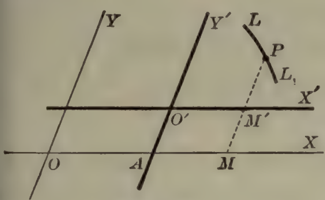

FIG. 56.

Draw $MM'P$ parallel to the y-axis; then

$$OM = OA + AM = OA + O'M',$$

i.e., and similarly,

$$\left.\begin{aligned} \boldsymbol{x} &= \boldsymbol{x' + h}, \\ \boldsymbol{y} &= \boldsymbol{y' + k}; \end{aligned}\right\} \qquad [26]$$

which are equations for *translating* the axes to the new origin.

These formulas are, of course, independent of the angle between the axes.

* For a more complete treatment of this subject see Tanner and Allen, Analytic Geometry, unabridged edition, Chapter VI.

As an illustration of the usefulness of such a change of axes, suppose there is given the equation

$$x^2 - 2hx + y^2 - 2ky = a^2 - h^2 - k^2, \tag{1}$$

referred to the axes OX and OY.

Now let $P \equiv (x, y)$ be any point on the locus L_1L of this equation, and let (x', y') be the same point P referred to the axes $O'X'$ and $O'Y'$, so that

$$x = x' + h \quad \text{and} \quad y = y' + k.$$

Substituting these values in the given equation gives:

$$(x' + h)^2 - 2h(x' + h) + (y' + k)^2 - 2k(y' + k) = a^2 - h^2 - k^2,$$

which reduces to

$$x'^2 + y'^2 = a^2;$$

a much simpler equation than (1), but representing the same locus, merely referred to other axes.

EXERCISES

1. What is the equation for the locus of $2x - 3y = 10$, if the origin be changed to the point $(^-1, ^-4)$, — direction of axes unchanged?

2. What does the equation $x^2 + y^2 + 10x - 6y + 4 = 0$ become if the origin be changed to the point $(^-5, 3)$, — direction of axes unchanged?

3. What does the equation $y^2 - 4x^2 + 2y + 8x - 7 = 0$ become when the origin is removed to $(1, ^-1)$, — direction of axes unchanged?

4. Find the equation for the straight line $y = mx + b$ when the origin is removed to the point $(0, b)$, — direction of axes unchanged.

5. Construct appropriate figures for exercises 1 and 4.

74. Transformation from one system of rectangular axes to another system, also rectangular, and having the same origin: change of direction of axes.

Let OX and OY be a given pair of rectangular axes, with any point $P \equiv (x, y)$; and let OX' and OY' be a second pair, with reference to which the same point is $P \equiv (x', y')$. Let $\theta = \angle XOX'$; hence, also $\theta = \angle YOY'$.

By Art. 13,

proj. $OP =$ proj. $OM' +$ proj. $M'P$.

Therefore, projecting upon OX,

$$x = x' \cos\theta + y' \cos(90° + \theta);$$

also, projecting upon OY,

$$y = x' \cos(90° - \theta) + y' \cos\theta:$$

and these equations may be written

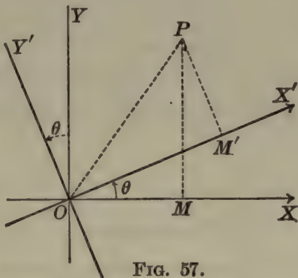

FIG. 57.

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}; \qquad [27]$$

which are formulas for *rotating* the axes through the angle θ.

It is clear that a combination of formulas [26] and [27] will transform from one pair of rectangular axes to any other pair of rectangular axes.

EXERCISES

Turn the axes through an angle of 45°, and find the new equations for the following loci:

1. $x^2 + y^2 = a^2$. 2. $x^2 - y^2 = 25$. 3. $y = -x + 10$.

4. $17x^2 - 16xy + 17y^2 = 225$.

5. If the axes are turned through the angle $\tan^{-1} 2$, what does the equation $4xy - 3x^2 = a^2$ become?

75. The degree of an equation in rectangular coördinates is not changed by transformation to other axes. Each formula of trans-

formation obtained, [26] and [27], has replaced x and y, respectively, by expressions of the first degree in the new coördinates x', y'. Therefore any one of these transformations replaces the terms containing x and y by expressions of the same degree, and so cannot *raise* the degree of the given equation. Neither can any one of these transformations *lower* the degree of the given equation; for if it did, then a transformation back to the original axes (which must give again the original equation) would *raise* the degree, which has just been shown to be impossible; hence these transformations leave the degree of an equation unchanged.

II. POLAR COÖRDINATES

76. Transformations between polar and rectangular systems. (1) *Transformation from a rectangular to a polar system, and vice versa, the origin and x-axis coinciding respectively with the pole and the initial line.* Let OX and OY be a given set of rectangular axes, and let OX and O be the initial line and pole for the system of polar coördinates. Also let P, any point in the plane, be (x, y) when referred to the rectangular axes, and (ρ, θ) in the polar system (Fig. 58); then

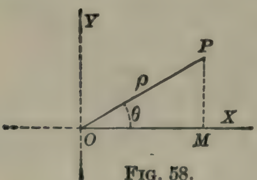

FIG. 58.

$$OM = OP \cos XOP;$$

i.e., $$x = \rho \cos \theta;$$
similarly, $$y = \rho \sin \theta. \qquad [28]$$

These are the required formulas of transformation when, *but only when*, the rectangular and polar axes are related as above described.

Conversely, from formulas [28], or directly from Fig. 58,

$$\rho = \sqrt{x^2 + y^2}, \ \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad [29]$$

which are the required formulas of transformation from polar to rectangular axes, under the above conditions.

(2) *Same as* (1) *except that the initial line OR makes an angle* α *with the x-axis.* It is at once evident that the formulas of transformation for this case are:

$$\left.\begin{aligned} x &= \rho\cos(\theta+\alpha), \\ \text{and} \quad y &= \rho\sin(\theta+\alpha). \end{aligned}\right\} \quad [30]$$

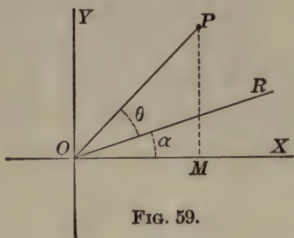

FIG. 59.

The converse formulas for this case are:

$$\rho = \sqrt{x^2+y^2}$$

$$\text{and} \quad \theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \alpha \quad = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \alpha. \quad [31]$$

A combination of formulas [26], [28], and [30] will transform from a set of rectangular axis to any polar set.

EXERCISES

Change the following to the corresponding polar equations, and draw a figure showing the two related systems of axes in each case. Take the pole at the origin, the polar axis coincident with the axis of x, in exercises 1 to 4.

1. $4x^2 + 4y^2 = a^2$.
2. $y^2 - x + 4ay = 0$.
3. $x^2 + y^2 = 25(x^2 - y^2)$.
4. $my = nx$.
5. $x - \sqrt{3}y + 2 = 0$, taking pole at origin, polar axis making the angle 60° with the x-axis.
6. $x^2 - y^2 - 4x - 6y - 54 = 0$, taking the pole at the point $(2, {}^-3)$, and the polar axis parallel to the x-axis.

Change the following to corresponding rectangular equations. Take the origin at the pole and the x-axis coincident with the polar axis.

7. $\rho = a.$

8. $\rho^2 \cos 2\theta = 4.$

9. $\rho^2 \sin 2\theta = 20.$

10. $\rho^2 = a^2 \sin 2\theta.$

SUGGESTION. In Ex. 10 multiply by ρ^2 and substitute $2 \sin\theta \cos\theta$ for $\sin 2\theta$; the equation then becomes $\rho^4 = 2a^2\rho^2 \sin\theta\cos\theta.$

11. $\rho = 10\cos\theta.$ **12.** $\theta = 3\tan^{-1}2.$ **13.** $\rho^{\frac{1}{2}}\cos\dfrac{\theta}{2} = k^{\frac{1}{2}}.$

EXAMPLES ON CHAPTER VII

1. Find the equation of the locus of $2xy - y + 6x - 11 = 0$ referred to parallel axes through the point $(\frac{1}{2}, {}^{-}3)$.

2. Transform the equation $x^2 - 4xy + 4y^2 - 6x + 12y = 0$ to new rectangular axes making an angle $\tan^{-1}\frac{1}{2}$ with the given axes.

3. Transform $x^2 - x - xy + 2y - 2 = 0$ to parallel axes through the point (2, 3). Draw an appropriate figure.

4. Transform the equation of Ex. 3 to axes bisecting the angles between the old axes. Trace the locus.

5. To what point must the origin be moved (the new axes being parallel to the old) in order that the new equation of the locus $2x^2 - 5xy - 3y^2 - 2x + 13y - 12 = 0$ shall have no terms of first degree?

SOLUTION. Let the new origin be (h, k); then $x = x' + h,\ y = y' + k$, and the new equation is

$$2(x'+h)^2 - 5(x'+h)(y'+k) - 3(y'+k)^2 - 2(x'+h) + 13(y'+k) - 12 = 0,$$

i.e.,

$$2x'^2 - 5x'y' - 3y'^2 + (4h - 5k - 2)x' - (5h + 6k - 13)y' + 2h^2 - 5hk - 3k^2 - 2h + 13k - 12 = 0;$$

but it is required that the coefficients of x' and y' shall be 0; *i.e.*, h and k are to be determined so that

$$4h - 5k - 2 = 0,$$

and $$5h + 6k - 13 = 0;$$

hence $$h = \tfrac{11}{7} \text{ and } k = \tfrac{6}{7}.$$

Therefore the new origin must be at the point $(\frac{11}{7}, \frac{6}{7})$, and the new equation is

$$2x'^2 - 5x'y' - 3y'^2 - 8 = 0.$$

6. The new axes being parallel to the old, determine the new origin so that the new equation of the locus

$$x^2 + 2xy - y^2 + 8x + 4y - 8 = 0$$

shall have no terms of first degree.

7. Transform the equations $x + y - 4 = 0$ and $3x - 2y + 5 = 0$ to parallel axes having the point of intersection of these lines as origin.

8. Transform the equation $\frac{x}{6} - \frac{y}{4} = 1$ to new rectangular axes through the point $(3, {}^-2)$, and making the angle $\tan^{-1}(\frac{2}{3})$ with the old axes.

9. Through what angle must the axes be turned that the new equation of the line $3x + y - 10 = 0$ shall have no y-term? Show this geometrically, from a figure.

10. Through what angle must the axes be turned in order that the new equation of the line $3x + 2y = 12$ shall have no x-term? Show analytically.

Solution. Let θ be the required angle; then the equations of transformation are

$$x = x'\cos\theta - y'\sin\theta \text{ and } y = x'\sin\theta + y'\cos\theta;$$

and the new equation is

$$(6\cos\theta + 4\sin\theta)x' - (6\sin\theta - 4\cos\theta)y' = 24;$$

but it is required that the coefficient of x be 0,

$$\therefore \qquad 6 \cos \theta + 4 \sin \theta = 0, \quad i.e., \quad \tan \theta = -\tfrac{6}{4};$$

whence $$\theta = \tan^{-1}(-\tfrac{3}{2}),$$

and the equation becomes

$$(6 \sin \theta - 4 \cos \theta) y' + 24 = 0,$$

which reduces to $$\frac{26}{\sqrt{13}} y' + 24 = 0,$$

i.e., to $$\sqrt{13}\, y' + 12 = 0.$$

11. Through what angle must the axes be turned to remove the x-term from the equation of the locus $Ax + By + C = 0$? to remove the y-term?

12. Show that to remove the xy-term from the equation of the locus, $2x^2 - 5xy - 3y^2 = 8$ [cf. Ex. 5], the axes must be turned through the angle $\theta = 67° 30'$, *i.e.*, so that $\tan 2\theta = -1$. What is the new equation?

13. Through what angle must a pair of rectangular axes be turned that the new x-axis may pass through the point $(2, 5)$?

14. What point must be the new origin, the direction of axes being unchanged, in order that the new equation of the line $Ax + By + C = 0$ shall have no constant term?

15. To what point, as origin of a pair of parallel axes, must a transformation of axes be made in order that the new equation of the locus, $x^2 - 2xy - 2x + 4y = 0$, shall have no terms of first degree? Construct the locus.

16. Find the new origin, the direction of axes remaining unchanged, so that the new equation of the locus, $x^2 + xy - x - 3y + 4 = 0$, shall have no constant term. Construct the figure.

17. Transform the equation $5x^2 + 2xy + 5y^2 = 12$ to new rectangular axes making an angle of $-45°$ with the given axes, — origin unchanged.

18. Transform $y^2 = 4x$ to new rectangular axes having the point $(9, 6)$ as origin, and making an angle $\tan^{-1}\frac{1}{3}$ with the old.

19. Transform to rectangular coördinates, the pole and initial line being coincident with the origin and x-axis, respectively:

(α) $\rho^2 = a^2 \cos 2\theta$, (β) $\rho^2 \cos 2\theta = a^2$, (γ) $\rho = k \sin 2\theta$.

Transform to polar coördinates, the x-axis and initial line being parallel:

20. $(x^2 + y^2)^2 = k^2(x^2 - y^2)$, the pole being at the point $(0, 0)$.

21. $x^2 - y^2 - 4x - 6y - 54 = 0$, pole being at the point $(2, {}^-3)$.

22. $x^2 + y^2 = 8ax$, the pole being at the point $(4a, 0)$.

23. Transform the equation $y^2 + 4ay \cot 30° - 4xy = 0$ to an oblique system of coördinates, with the same origin and x-axis, but the new y-axis at an angle of $30°$ with the old x-axis.

24. Prove the formula for the distance in polar coördinates [21], p. 118, by transformation of the corresponding formula [1], p. 24, in rectangular coördinates.

25. Transform the equation $x \cos \alpha + y \sin \alpha = p$ to polar coördinates.

26. Given oblique axes making the angle $X'OY' = \omega$, and a second set, of rectangular axes, XOY, with the same origin and

$$\angle XOX' = \theta, \qquad \angle XOY' = \phi,$$

show by projection that

$$x = x' \cos \theta + y' \cos \phi \qquad y = x' \sin \theta + y' \sin \phi.$$

CHAPTER VIII

THE CONIC SECTIONS

EQUATION OF THE SECOND DEGREE

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0.^*$$

77. Recapitulation. The position of a point may be represented by numbers — its coördinates [Chapter II] — and therefore a geometric curve may be represented by an algebraic equation [Chapter III], and the properties of the curve studied by means of the equation [Chapters III, IV, V]. The fundamental relation is that the coördinates of every point of the curve, and of no other point, shall satisfy the equation. For a given point, the values of the coördinates will depend upon the system chosen — rectangular, or polar, etc. — and also upon the particular origin and axes chosen [Chapters VI, VII]. Hence, for a given curve, the corresponding equation will depend upon the system of coördinates chosen, and also upon the particular origin and axes. But [Art. 75] the *degree* of the equation in a chosen system does not depend upon the particular axes taken for reference, but upon the nature of the curve. For example, in rectangular coördinates the equation of the first degree represents a straight line [Chapter IV]. In taking up the general study of loci, it is natural to classify

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* See Art. 153, general equation of second degree with three variables, to compare coefficients with those used here.

the loci according to the degree of their equations. Equations of the first degree having been studied, the various equations of the second degree will now be taken.

It has, however, been found convenient with the straight line and circle to find, first, simple standard equations of the loci [equations 8–12, 16, and 17], to study the curves by means of these, and then to reduce more complicated equations to these familiar forms. This method of work is, in fact, what gives analytic geometry much of its usefulness. It can be used for equations of the second degree, since, happily, each represents some **conic section**, *i.e.*, one of the curves of intersection of a plane and a right circular cone. These loci are the ellipse, the parabola, and the hyperbola (with the circle and straight line as special cases); they are of primary importance in astronomy, where it is found that the orbits of the earth, the comets, and other heavenly bodies are curves of this kind.

78. The conic sections. The conic sections, or more briefly, conics, may be defined thus:

A conic is a plane curve traced by a point which moves so that its distance from a fixed point bears always a constant ratio to its distance from a fixed straight line. The general equation of this locus will first be derived, then from it the various special standard forms will be obtained.

(*a*) *The equation of the locus.* Let F be the fixed point, — the **focus** of the curve; $D'D$ the fixed line, — the **directrix** of the curve; and e the given ratio, — the **eccentricity** of the curve.

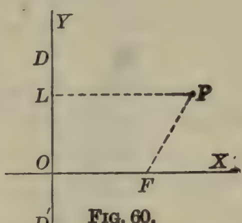

FIG. 60.

Let $D'D$ be the y-axis, and the perpendicular to it through F, *i.e.*, the line OFX, be the x-axis. Let $P \equiv (x, y)$ be any position of the generating point,

and let $OF = k$ be the fixed distance of the focus from the directrix; then $F \equiv (k, 0)$. Draw FP, also $LP \perp D'D$.

Then $$FP : LP = e, \qquad \text{[geom. eq.]}$$

but $FP = \sqrt{(x-k)^2 + y^2}$, and $LP = x$,

hence $$\sqrt{(x-k)^2 + y^2} = ex; \qquad \text{[alg. eq.]}$$

i.e., $$(1-e^2)x^2 + y^2 - 2\,kx + k^2 = 0, \tag{1}$$

which is the equation of the given locus.

This equation is of the second degree; later it will be shown that *every* equation of the second degree between two variables represents a conic section.

(*b*) *Discussion of equation* (1).

Equation (1) shows that for every value of x, the two corresponding values of y are numerically equal but of opposite signs, hence *the conic is symmetrical with regard to the x-axis as here chosen.* For this reason the line drawn through the focus of a conic and perpendicular to the directrix is called the **principal axis** of the conic.

If $x = 0$, then $y = \pm k\sqrt{-1}$, which shows that this curve does not intersect the y-axis as here chosen; *i.e., a conic does not intersect its directrix.*

If $y = 0$, then $(1-e^2)\,x^2 - 2\,kx + k^2 = 0$,

whence $$x = \frac{k}{1+e}, \text{ or } x = \frac{k}{1-e}, \tag{2}$$

hence, *if $e \neq 1$, the conic meets the principal axis* (the x-axis as here chosen) *in two points.* These points are called the **vertices** of the conic.

If $e = 1$, the conic meets the principal axis in only one point at a finite distance from the directrix. This point is then spoken of as the **vertex**.

The *form* of the locus of equation (1) depends upon the value of the eccentricity, e; if $e = 1$, the conic is called a **parabola**; if

$e < 1$, an **ellipse**; and if $e > 1$, an **hyperbola.** Each of these cases will now be separately considered.

I. THE PARABOLA

Special Equation of Second Degree

$$\boldsymbol{Ax^2 + 2\,Gx + 2\,Fy + C = 0, \text{ or } By^2 + 2\,Gx + 2\,Fy + C = 0}$$

79. The parabola defined. A parabola is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line. It is the conic section with eccentricity $e = 1$.

The equation of a parabola, with any given focus and directrix, can be obtained directly from this definition.

EXAMPLE. To find the equation of the parabola whose directrix is the line $x - 2\,y - 1 = 0$, and whose focus is the point $(2, {}^-3)$.

Let $P \equiv (x, y)$ be any point on the parabola; then $\dfrac{x - 2\,y - 1}{-\sqrt{5}}$ is the distance of P from the directrix [Art. 51], and $\sqrt{(x-2)^2 + (y+3)^2}$ is the distance of P from the focus [Art. 19];

hence $$\frac{x - 2\,y - 1}{-\sqrt{5}} = \sqrt{(x-2)^2 + (y+3)^2},$$

by definition; that is,

$$4\,x^2 + 4\,xy + y^2 - 18\,x + 26\,y + 64 = 0;$$

which is the required equation.

The equation obtained in this way is not, however, in the most suitable form from which to study the properties of the curve, but can be simplified by a proper choice of axes.

If the line of symmetry is taken as the x-axis, the equation will have no y-term of first degree [cf. Art. 78, eq. (1)];

while if the vertex of the curve be taken as origin, the equation will have no constant term, since the point (0, 0) must satisfy the equation. With this choice of axes, the equation of the parabola will reduce to a simple form, the *first standard equation* of the parabola.

80. First standard form of the equation of the parabola. Let $D'D$ be the directrix of the parabola, and F its focus; also let ZFX, perpendicular to the directrix, be the x-axis; denote the fixed distance ZF by $2p$, and let O, its middle point, be the origin; then OY, perpendicular to OX, is the y-axis. Let $P \equiv (x, y)$ be any point on the curve, and draw LQP perpendicular to OY; also draw the ordinate MP, and the line FP. The line FP is called the **focal radius** of P.

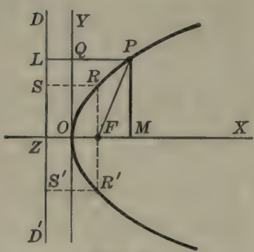

FIG. 61.

To apply the method of Art. 79, the equation of the directrix and the coördinates of the focus are needed.

Since $$ZO = OF = p,$$

the equation of the directrix is $x + p = 0$, (1)

and *the focus is the point* $(p, 0)$. (2)

Now, from the definition of the parabola,

$$FP = LP; \qquad \text{[geom. eq.]}$$

but $FP = \sqrt{(x-p)^2 + y^2}$, and $LP = ZO + OM = p + x$;

hence $$\sqrt{(x-p)^2 + y^2} = (x + p), \qquad \text{[alg. eq.]}$$

whence $$\boldsymbol{y^2 = 4\,px}, \qquad [32]$$

which is the desired equation.

This first standard form [32] is the simplest equation of the parabola, and the one which will be most used in the subsequent study of the curve. It will be seen later [Arts. 84, 98,

102] that any equation which represents a parabola can be reduced to this form.

81. To trace the parabola $y^2 = 4px$. From equation [32] it follows that (Art. 30):

(1) The parabola passes through the point *O*, halfway from the directrix to the focus, the **vertex** of the curve.

(2) The parabola is symmetrical with regard to the x-axis; *i.e.*, with regard to the line through the focus perpendicular to the directrix, the **axis*** of the curve.

(3) The abscissa x has always the same sign as the constant p, *i.e.*, that the entire curve and its focus lie on the same side of a line parallel to the directrix, and midway between the directrix and the focus.

(4) The abscissa x may vary in magnitude from 0 to ∞; and when x increases, so also does y (numerically); hence the parabola is an open curve, receding indefinitely from its directrix and its axis.

The parabola is, then, an open curve of one branch which lies on the same side of the directrix as does the focus; when constructed it has the form shown in Fig. 61.

82. Latus rectum. The chord through the focus of a conic, parallel to the directrix, is called its **latus rectum.** In Fig. 61 this chord is $R'R$.

Now $$R'R = 2\,FR = 2\,SR = 2\,ZF = 4\,p.$$

Hence *the length of the latus rectum of the parabola is* **4 *p*;** that is, it is *equal to the coefficient of x in the first standard equation.*

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* The *axis of a curve* should be carefully distinguished from an *axis of coördinates*; though they often are coincident lines in the figures to be studied.

83. Intrinsic property of the parabola. Second standard equation. Equation [32] may be interpreted as stating a geometric property of the parabola, — a property which belongs to every point of the parabola, whatever coördinate axes be chosen. For (see Fig. 61) the equation $y^2 = 4\,px$ gives the geometric relation

$$\overline{MP}^2 = 4\,OF \cdot OM = R'R \cdot OM,$$

or, expressed in words,

If from any point on the parabola, a perpendicular is drawn to the axis of the curve, the square on this perpendicular is equivalent to the rectangle formed by the latus rectum and the line from the vertex to the foot of the perpendicular.

This geometric property enables one to write down immediately the equation of the parabola, whenever the axis of the curve is parallel to one of the coördinate axes.

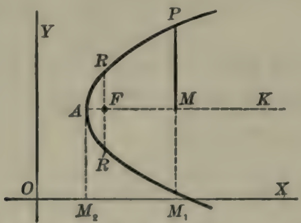

FIG. 62.

E.g., if the vertex of the parabola is the point $A \equiv (h, k)$, and its axis is parallel to the x-axis, as in the figure, F being the focus and $P \equiv (x, y)$ any point on the parabola. Draw MP perpendicular to the axis AK. Then

$$\overline{MP}^2 = 4\,AF \cdot AM, \quad i.e, \quad (y-k)^2 = 4\,p(x-h), \qquad [33]$$

which is the equivalent algebraic equation. This may be taken as a *second standard form* of the equation; it represents the parabola whose vertex is at the point (h, k), whose axis is parallel to the x-axis, and, if p is positive, which lies wholly on the positive side of the line $x = h$.

Equation [33] evidently may be reduced to equation [32] by a transformation of coördinates to parallel axes through the vertex as the new origin.

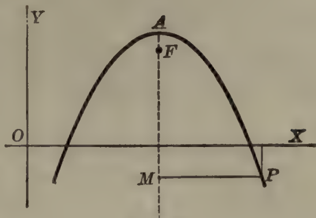

FIG. 63.

Again, if the vertex is

$$A \equiv (h, k)$$

and the axis is parallel to the y-axis, then the equation is

$$(x - h)^2 = 4\,p(y - k), \qquad [34]$$

which is another form for the second standard equation of the parabola. (In Fig. 63, p is negative.)

EXERCISES

Construct and find the equations of parabolas:

1. With the focus at the point $(^-3, {}^-3)$ and with the line $x + y = 2$ as directrix.

2. With the focus at the origin, and with the line $y - 2x + 3 = 0$ as directrix;

3. With the vertex at the origin, and the focus at the point $(4, 0)$;

4. With the vertex at the origin, and the focus at the point $(0, {}^-a)$;

5. With the vertex at the point $(^-2, 5)$, and the focus at the point $(^-2, 1)$;

6. With the vertex at the point $(^+2, {}^+4)$, and the focus at the point $(^-1, {}^+4)$;

7. With the focus at the point $(0, 2\,p)$ and the line $y = 0$ as directrix.

8. What is the latus rectum of each of the parabolas of exercises 3 to 6?

9. Describe the effect produced on the form of a parabola by increasing or decreasing the length of its latus rectum.

84. Every equation of the form $Ax^2 + 2Gx + 2Fy + C = 0$, or $By^2 + 2Gx + 2Fy + C = 0$, represents a parabola whose axis is parallel to one of the coördinate axes.

Equations [32], [33], and [34] are of the form

$$By^2 + 2Gx + 2Fy + C = 0, \text{ or } Ax^2 + 2Gx + 2Fy + C = 0;$$

that is, each has one and only one term containing the square of a variable, and no term containing the product of the two variables. Conversely, it may be shown that an equation of either of these forms represents a parabola whose axis is parallel to one of the coördinate axes.

A numerical example will indicate the method of proof [cf. Art. 56]; it is analogous to "completing the square" in the solution of quadratic equations.

EXAMPLE. Given the equation

$$25y^2 - 30y - 50x + 89 = 0,$$

to show that it represents a parabola; and to find its vertex, focus, and directrix.

Divide both members of the equation by 25, and complete the square of the y-terms; the equation may then be written

$$y^2 - \tfrac{6}{5}y + \tfrac{9}{25} = 2x - \tfrac{89}{25} + \tfrac{9}{25},$$

that is, $$(y - \tfrac{3}{5})^2 = 2(x - \tfrac{8}{5}),$$

whence $$(y - \tfrac{3}{5})^2 = 4 \cdot \tfrac{1}{2} \cdot (x - \tfrac{8}{5}).$$

Now this equation is in the second standard form (cf. equation [33]), and therefore every point on its locus has the geometric property given in Art. 83;

and the locus is a parabola. The vertex is at the point $(\frac{8}{5}, \frac{3}{5})$; the axis is parallel to the x-axis, extending in the positive direction; and, since $p = \frac{1}{2}$, the focus is at the point $(\frac{21}{10}, \frac{3}{5})$, and the directrix is the line $x = \frac{11}{10}$.

Now by the same method the more general equation, viz.,

$$Ax^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

may be reduced to the form

$$\left(x + \frac{G}{A}\right)^2 = 4\left(-\frac{F}{2\,A}\right)\left(y - \frac{G^2 - AC}{2\,AF}\right) \tag{2}$$

which is of the standard form [34] of a parabola.

NOTE. The transformation just given fails if $A = 0$ or if $F = 0$, for in that case some of the terms in the last equation are infinite. If, however, $A = 0$, the given equation becomes $2\,Gx + 2\,Fy + C = 0$; and, this being of the first degree, represents a straight line. If, on the other hand, $F = 0$, the given equation reduces to $Ax^2 + 2\,Gx + C = 0$, and represents two straight lines each parallel to the y-axis; they are real and distinct, real and coincident, or imaginary, depending upon the value of $G^2 - AC$. These lines may be regarded as limiting forms of the parabola.

EXERCISES

Determine the vertex, focus, latus rectum, equation of the directrix and of the axis for each of the following parabolas; also sketch each of the figures:

1. $y^2 - 4\,y = -4\,x + 8$;
2. $3\,x^2 + 6\,x - y + 7 = 0$;
3. $5\,y - 1 = 3\,y^2 + 4\,x$;
4. $4\,y^2 - 8\,ay - x + 4\,a^2 = b$;
5. $Ax^2 + 2\,Gx + 2\,Fy + C = 0$;
6. $By^2 + 2\,Gx + 2\,Fy + C = 0$.

7. Derive equation [32] as a special case of equation (1), Art. 78.

8. Reduce the equation $4\,x^2 + 4\,xy + y^2 - 18\,x + 26\,y + 64 = 0$ [cf. Art. 79] to form [32] by translating axes to the point $(2, -3)$ as origin, then rotating the axes through the angle $\theta = \tan^{-1}\frac{1}{2}$.

II. THE ELLIPSE

Special Equation of the Second Degree

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0.$$

85. The ellipse defined. An ellipse is the locus of a point which moves so that the ratio of its distance from a fixed point, to its distance from a fixed line, is constant and less than unity. This curve is the conic section with eccentricity $e < 1$.

The equation of an ellipse with any given focus, directrix, and eccentricity may be readily obtained from this definition.

Example. An ellipse of eccentricity $\frac{2}{3}$ has its focus at $(2, {}^-1)$, and has the line $x + 2\,y = 5$ for directrix: To find its equation. Let $P \equiv (x, y)$ be any point on the curve, F the focus, and PQ the perpendicular from P to the directrix.

Then $\qquad FP = \frac{2}{3}\,QP;$

but $\qquad FP = \sqrt{(x-2)^2 + (y+1)^2},$

$$QP = \frac{x + 2\,y - 5}{+\sqrt{5}} \qquad \text{[Arts. 19, 51]},$$

hence $\quad (x-2)^2 + (y+1)^2 = \frac{4}{45}(x + 2\,y - 5)^2;$

that is, $\quad 41\,x^2 - 16\,xy + 29\,y^2 - 140\,x + 170\,y + 125 = 0;$

which is the equation of the given ellipse.

To obtain the simplest form of the equation, more suitable for studying the properties of the curve, the line of symmetry should be chosen as the x-axis, with the origin halfway between the two vertices. [Art. 78 (b)]. The resulting equation is the *first standard form* of the equation of the ellipse. It is derived in the next article.

86. The first standard equation of the ellipse. Let F be the focus, $D'D$ the directrix, and ZFX the perpendicular to $D'D$

through F, cutting the curve in the two points A' and A. Denote by $2a$ the length of AA', and let O be its middle point, making

$$AO = OA' = a;$$

and let ZX be the x-axis, O the origin, and OY, perpendicular to OX, the y-axis.

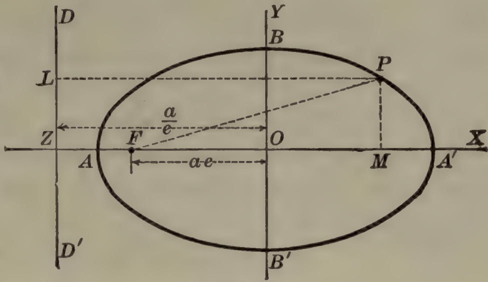

FIG. 64.

Now the equation of the directrix and the coördinates of the focus are needed, so that the method of Art. 85 may be applied

For the directrix:

$$ZO = ZA + AO;$$

but $\qquad 2(ZA + AO) = ZA + (ZA + AA') = ZA + ZA',$

and, by definition of the ellipse,

$$AF = eZA \quad \text{and} \quad FA' = eZA',$$

hence $\qquad 2(ZA + AO) = \dfrac{AF + F'A}{e} = \dfrac{AA'}{e} = \dfrac{2a}{e}.$

Therefore, $\qquad ZO = \dfrac{a}{e},$

and *the equation of the directrix is* $\quad x + \dfrac{a}{e} = 0. \qquad (1)$

For the focus: $\quad FO = AO - AF = FA' - FO - AF.$

$\therefore \qquad 2FO = FA' - AF = e(ZA' - ZA),$

i.e., $\qquad = eAA' = 2ae,$

therefore $\qquad FO = ae,$

and *the focus F is the point* $\quad (^{-}ae, 0). \qquad (2)$

For the equation of the ellipse: Let $P \equiv (x, y)$ be any point on the curve, and draw $LP \perp D'D$;

then $$FP = eLP;\qquad\text{[geom. eq.]}$$

but $$FP = \sqrt{(x+ae)^2 + y^2},\qquad LP = \frac{a}{e} + x;$$

hence $$(ae+x)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,\qquad\text{[alg. eq.]}$$

that is, $$(1-e^2)x^2 + y^2 = a^2(1-e^2),$$

that is, $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \tag{3}$$

From equation (3), the intercepts of the curve on the y-axis are $\pm a\sqrt{1-e^2}$. Both intercepts are real, since $e < 1$; hence the ellipse cuts the y-axis in two real points B and B', on opposite sides of the origin O, and equidistant from it. If OB is denoted by b, so that

$$b^2 = a^2(1-e^2), \tag{4}$$

equation (3) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.* \qquad [35]$$

This is the simplest equation of the ellipse. As will be seen later, every equation representing an ellipse can be reduced to this form.

87. To trace the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. From equation [35] it follows that:

(1) The ellipse is symmetrical with regard to the x-axis; *i.e.*, with regard to the line through the focus and perpendicular to the directrix, the **principal axis** of the curve;

---

* If $a = b$ (*i.e.*, if $e = 0$) this equation represents a circle. The ellipse, then, includes the circle as a special case. In other words: a circle is an ellipse whose eccentricity is zero

(2) The ellipse is symmetrical with regard to the y-axis also; *i.e.*, with regard to a line parallel to the directrix and halfway between the vertices (Fig. 65);

(3) For every value of x from $-a$ to $+a$, the two corresponding values of y are real, equal numerically, but opposite in sign; and for every value of y from $-b$ to b, the two values of x are real and equal numerically, but opposite in sign; and that neither x nor y can have real values beyond these limits.

The ellipse is, therefore, a closed curve, of one branch, which lies wholly on the same side of the directrix as the focus; and the curve has the form represented in Fig. 65.

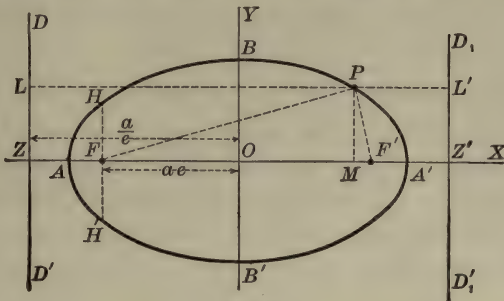

FIG. 65.

The segment AA' (Fig. 65) of the principal axis intercepted by the curve is called its **major** or **transverse axis**; the corresponding segment $B'B$ is its **minor** or **conjugate axis**. From the symmetry of the curve with respect to these axes it follows that it is also symmetrical with respect to their intersection O, the **center** of the ellipse.

From this symmetry about the y-axis it follows also that the ellipse has a second focus, $F' \equiv (ae, 0)$ (Fig. 65) and a second directrix D'_1D_1 on the positive side of the minor axis, whose equation is $x - \frac{a}{e} = 0$.

The **latus rectum** of the ellipse, *i.e.*, the focal chord parallel to the directrix [Art. 82], is evidently twice the ordinate of the point whose abscissa is ae. But if $x_1 = ae$, $y_1 = b\sqrt{1-e^2}$; or, since $b = a\sqrt{1-e^2}$, $y_1 = \frac{b^2}{a}$.

Hence, *the latus rectum* is $\frac{2\,b^2}{a}$.

88. Intrinsic property of the ellipse. Second standard equation. Equation [35] states a geometric property which belongs to every point of the ellipse, whatever the coördinate axes chosen, and to no other point: viz., if P be any point of the ellipse (Fig. 65), then

$$\frac{\overline{OM}^2}{\overline{OA'}^2} + \frac{\overline{MP}^2}{\overline{OB}^2} = 1;$$

i.e., stated in words:

If from any point on the ellipse a perpendicular be drawn to the transverse axis; then the square of the distance from the center of the ellipse to the foot of this perpendicular, divided by the square of the semi-transverse axis, plus the square of the perpendicular divided by the square of the semi-conjugate axis, equals unity.

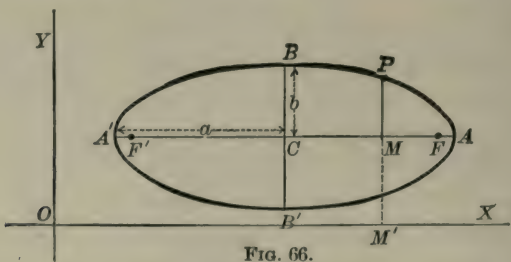

Fig. 66.

This geometric or physical property belongs to no point not on the curve, and therefore completely determines the ellipse. It enables one to write immediately the equation of any ellipse whose axes are parallel to the coördinate axes.

For example: if, as in Fig. 66, the major axis of an ellipse is parallel to the x-axis, and the center is at the point $C \equiv (h, k)$, let $P \equiv (x, y)$ be any point on the curve, and a, b be the semi-axes, then

$$\frac{\overline{CM}^2}{\overline{CA}^2} + \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$

that is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \qquad [36\,a]$$

which is the equation of the given ellipse.

Or again, if, as in Fig. 67, the major axis is parallel to the y-axis; then,

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \qquad [36\,b]$$

which is the equation of the given ellipse.

Equations [36 a and b] may be considered *second standard* forms of the equation of the ellipse; by a change of coördinates to a set of parallel axes through the center $C \equiv (h, k)$, as the new origin, either can be reduced to the first standard form.

By Art. 86 the distance from the center of an ellipse to its focus is ae; but since $b^2 = a^2(1 - e^2)$* [Art. 86, equation (4)], therefore $ae = \sqrt{a^2 - b^2}$; hence, in Figs. 66 and 67,

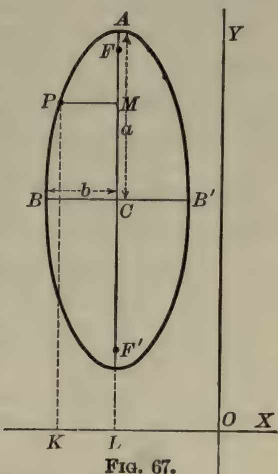

FIG. 67.

* The student should observe that b is the *semi-minor axis* and not necessarily the denominator of y^2 in the standard forms of the equation of the ellipse — [35] and [36 a] and [36 b]; he should also observe that the foci are always on the *major axis*.

$$F'C = CF = ae = \sqrt{a^2 - b^2}.$$

Again, the equation of an ellipse, in either standard form, gives the semi-axes as well as the center of the curve, therefore the positions of the foci are readily determined from either standard form of the equation.

EXERCISES

Construct the following ellipses, and find their equations:

1. Given the focus at the point (3, 3), the equation of the directrix $x + y - 3 = 0$, and the eccentricity $\frac{1}{2}$;

2. Given the focus at the origin, the equation of the directrix $x = 9$, and the eccentricity $\frac{1}{2}$;

3. Given the focus at the point (0, 1), the equation of the directrix $y - 8 = 0$, and the eccentricity $\frac{1}{4}$;

4. Given the center at the origin, and the semi-axes $\sqrt{3}$, $\sqrt{7}$. Find also the latus rectum.

Find the equation of an ellipse whose center is at the origin, whose axes are the coördinate axes:

5. Which passes through the two points $(^-2, ^-2)$ and $(^-3, ^+1)$? How many such ellipses are there?

6. Whose foci are the points (3, 0), $(^-3, 0)$, and eccentricity $\frac{1}{3}$;

7. Whose foci are the points (0, 6), $(0, ^-6)$, and eccentricity $\frac{2}{3}$;

8. Whose latus rectum is 5, and eccentricity $\frac{2}{3}$;

9. Whose latus rectum is 8, and minor axis 10;

10. Whose major axis is 10, and which passes through the point $(4, \frac{9}{5})$.

Draw the following ellipses, locate their foci, and find their equations:

11. Given the center at the point $(3, {}^{-}2)$, the semi-axes 4 and 3, and the major axis parallel to the x-axis;

12. Given the center at the point $(1, 8)$, the semi-axes 4 and 9, and the major axis parallel to the y-axis;

13. Given the center at the point $(0, 7)$, the origin at a vertex, and $(2, 3)$ a point on the curve;

14. Given the circumscribing rectangle, whose sides are the lines $x - a = 0$, $y - c = 0$, $x + b = 0$, $y + d = 0$; the axes of the curve being parallel to the coördinate axes.

15. If b becomes more and more nearly equal to a, what curve does the ellipse approach as a limit? What is the limit of e? of the directrices?

89. Every equation of the form $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$, in which A and B have the same sign, represents an ellipse whose axes are parallel to the coördinate axes. Equations [35] and [36], obtained for the ellipse, are all, when expanded, of the form

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

where *A and B have the same sign, and neither of them is zero.* Conversely, an equation of this form represents an ellipse the axes of which are parallel to the coördinate axes. A numerical example will suggest the general proof.

Example. Given the equation $4\,x^2 + 9\,y^2 - 16\,x + 18\,y - 11 = 0$, to show that it represents an ellipse, and to find its elements. On completing the square for the terms in x, and also for those in y, and transposing, this equation becomes

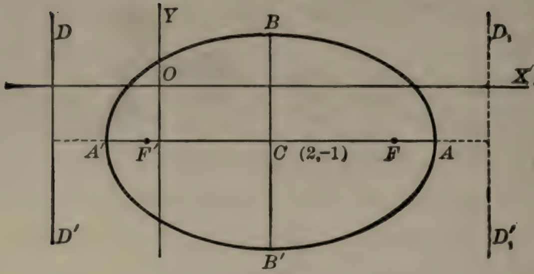

Fig. 68.

$$4x^2 - 16x + 16 + 9y^2 + 18y + 9 = 11 + 16 + 9,$$

that is $$4(x-2)^2 + 9(y+1)^2 = 36;$$

hence $$\frac{(x-2)^2}{3^2} + \frac{(y+1)^2}{2^2} = 1.$$

This equation is of the form [36], and, therefore, its locus has the geometric property given in Art. 88, and is an ellipse. Its center is the point $(2, -1)$; its major axis is parallel to the x-axis, of length 6; its minor axis is of length 4; the foci are the points

$$F' = (2 - \sqrt{5}, -1), \qquad F = (2 + \sqrt{5}, -1);$$

and the equations of the directrices are, respectively,

$$x = 2 - \frac{9}{\sqrt{5}}, \qquad x = 2 + \frac{9}{\sqrt{5}}.$$

EXERCISES

Determine for each of the following ellipses the center, semi-axes, foci, vertices, and latus rectum; then sketch each curve:

1. $3x^2 + 9y^2 - 12x - 18y - 6 = 0.$
2. $4x^2 + y^2 - 8x + 2y + 1 = 0.$
3. $15x^2 + y^2 + 60x + 4y + 15 = 0.$
4. $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$

5. By completing the squares of the x-terms and of the y-terms, and a suitable transformation of coördinates, reduce the equations of exercises 1, 2, and 3 to the standard form [35].

III. THE HYPERBOLA

Special Equation of the Second Degree

$$Ax^2 - By^2 + 2\,Gx + 2\,Fy + C = 0$$

90. The hyperbola defined. An hyperbola is the locus of a point which moves so that the ratio of its distance from a fixed point, to its distance from a fixed line, is constant and greater than unity. This curve is the conic section with eccentricity $e > 1$.

91. The first standard form of the equation of the hyperbola. Let F be the focus, $D'D$ the directrix, the line of symmetry ZFX as the x-axis, and OY perpendicular to ZX halfway between the two vertices A and A' [see Art. 78] the y-axis. Let $2\,a$ be the distance $A'A$ between the vertices.

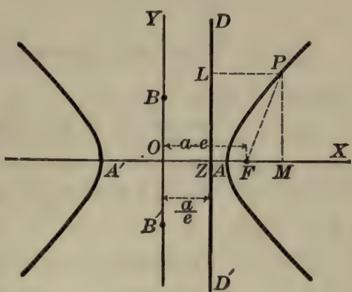

FIG. 69.

Since the hyperbola differs from the ellipse only in the relative values of e and 1, *i.e.*, in the sign of $1 - e^2$, which is $+$ in the ellipse and $-$ in the hyperbola, the standard equation of the hyperbola can be derived by the method of Art. 86, and it will be found that with choice of axes and notation as there given, the results given in equations (1), (2), and (3) of that article apply equally to the hyperbola. If now, since $1 - e^2$ is negative, the substitution $b^2 = a^2(e^2 - 1)$ is made, equation (3), Art. 86, will

become $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad [37]$$

the standard equation of the hyperbola. Every equation representing an hyperbola can be reduced to this form, as is shown later.

Here, as in Art. 86,

$$OF = ae,$$

therefore *the focus F is the point* $(ae, 0)$. (4)

Similarly for the directrix:

$$OZ = \frac{a}{e},$$

hence *the directrix is the line* $x - \frac{a}{e} = 0.$ (5)

As above defined, b is real, and its value is known when a and e are known. In Fig. 69,

$$OB = b, \quad OB' = -b, \quad \text{and} \quad b = a\sqrt{e^2 - 1}. \qquad (6)$$

92. To trace the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Equation [37] shows that:

(1) The hyperbola is symmetrical with regard to the x-axis, *i.e.*, with respect to the line through the focus and perpendicular to the directrix, called the **principal axis** of the hyperbola.

(2) The hyperbola is symmetrical with regard to the y-axis also; *i.e.*, with regard to the line parallel to the directrix and halfway between the vertices.

(3) For every value of x from $-a$ to a, y is imaginary; while for every other value of x, y is real and has two values, equal numerically but opposite in sign. But for every value of y, x has two real values, equal numerically and opposite in sign. When x increases numerically from a to ∞, then y also increases numerically from 0 to ∞.

These facts show that no part of the hyperbola lies between the two lines perpendicular to its principal axis and drawn through the vertices of the curve; but that it has two open infinite branches, lying outside of these two lines. The form of the hyperbola is as represented in Fig. 70.

The segment $A'A$ of the principal axis, intercepted by the curve, is called its **transverse axis.** The segment $B'B$ of the second line of symmetry (the y-axis), where $B'O = OB = b$, is called the **conjugate axis**; and although not cut by the hyperbola, it bears important relations to the curve. From the symmetry of the hyperbola, with respect to these axes, it follows that it is also symmetrical with respect to their intersection O, the **center** of the curve.

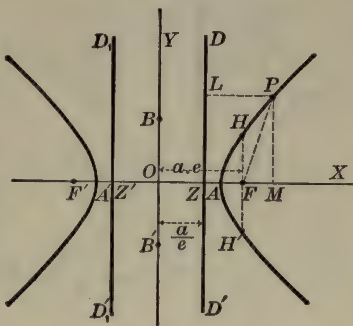

Fig. 70.

From this symmetry about the y-axis it follows also that there is a second focus, the point $F \equiv (^{-}ae, 0)$, and a second directrix on the negative side of the conjugate axis, whose equation is $x + \frac{a}{e} = 0$. [Cf. Art. 87].

The **latus rectum** of the hyperbola is readily found to be $\frac{2b^2}{a}$. [Cf. Arts. 82, 87].

93. Intrinsic property of the hyperbola. Second standard equation. Equation [37] states a geometric property which belongs to every point of an hyperbola, whatever the coördinate axes chosen, and to no other point; and which therefore completely defines the hyperbola. With the figure and notation of Art. 91 equation [37] states that (see Fig. 70),

$$\frac{\overline{OM}^2}{\overline{OA}^2} - \frac{\overline{MP}^2}{\overline{OB}^2} = 1,$$

a property entirely analogous to that of Art. 88 for the ellipse. It enables one to write at once the equation of an hyperbola with given center and semi-axes, — the axes of the curve being parallel to the coördinate axes.

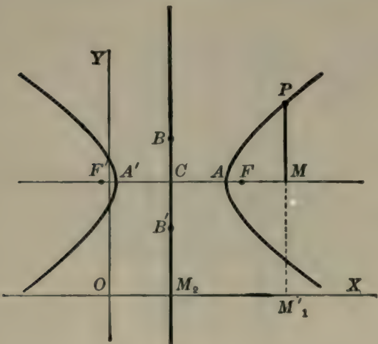

FIG. 71.

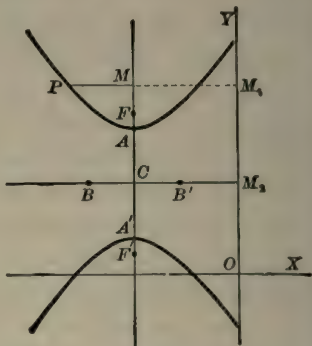

FIG. 72.

For example, if the transverse axis is parallel to the x-axis, as in Fig. 71, and the center at the point $C \equiv (h, k)$, and if $P \equiv (x, y)$ is any point on the curve, then

$$\frac{\overline{CM}^2}{\overline{CA}^2} - \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$

i.e.,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \qquad [38\,a]$$

which is the equation of the hyperbola, with a and b as semi-axes.

Again, if the transverse axis is parallel to the y-axis, as in Fig. 72, with the center at the point (h, k), the equation of the hyperbola will be found to be

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$$

i.e.,
$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1. \qquad [38\ b]$$

NOTE 1. That the expressions obtained on p. 154 for the distances from the center to the focus and the directrix, of hyperbola [37], are equally true for hyperbolas [38 a] and [38 b] follows from the fact that those expressions involve only a, b, and e; moreover, equation (6) of Art. 91 determines e in terms of a and b; hence, for all these hyperbolas $e^2 = \frac{a^2+b^2}{a^2}$, the distances from the center to the foci are given by

$$ae = \pm\sqrt{a^2+b^2},$$

and those to the directrices by

$$\frac{a}{e} = \frac{a^2}{\pm\sqrt{a^2+b^2}}.$$

NOTE 2. It should be noticed that in equations [37], [38 a], [38 b], the negative term involves that one of the coördinates which is parallel to the conjugate axis.

EXERCISES

1. Find the equation of the hyperbola having its focus at the point ($^+1$, $^+1$), for its directrix the line $y=4$, and eccentricity 2. Plot the curve. [Cf. Art. 78, and Art. 84, Ex.].

Find the equation of the hyperbola having its center at the origin and

2. Semi-axes equal, respectively, to 3 and 5;

3. Conjugate axis 16, — the point (20, 5) being on the curve;

4. The distance between the foci 10, and eccentricity $\sqrt{\frac{5}{3}}$;

5. The distance between the foci equal to twice the transverse axis.

[Two solutions for Ex. 2, 3, 4, and 5.]

Find the equation of an hyperbola:

6. With center at the point (4, 5), semi-axes 5 and 6, and the transverse axis parallel to the x-axis; plot the curve;

7. With center at the point $(^-4, ^-5)$, semi-axes 3 and 2, and the transverse axis parallel to the y-axis. Plot the curve.

8. Find the foci and latus rectum for the hyperbolas of exercises 6 and 7.

9. By a suitable transformation of coördinates, reduce the equation of exercise 6 to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

10. Find the foci of the hyperbolas,

(α) $\frac{x^2}{5}-\frac{y^2}{4}=1$, (β) $\frac{x^2}{10}-\frac{y^2}{9}=1$, (γ) $\frac{y^2}{9}-\frac{x^2}{16}=1$.

Plot the curves (β) and (γ).

94. Every equation of the form $Ax^2+By^2+2\,Gx+2\,Fy+C=0$, in which A and B have unlike signs, represents an hyperbola whose axes are parallel to the coördinate axes. When cleared of fractions and expanded, the three equations found for the hyperbola are of the form

$$Ax^2+By^2+2\,Gx+2\,Fy+C=0, \tag{1}$$

where A and B have opposite signs, and neither of them is zero. Conversely, it may be shown that every equation of this form represents an hyperbola, whose axes are parallel to the coordinate axes. A numerical example will suggest the general proof.

Example: The equation

$$9\,x^2-4\,y^2-18\,x+24\,y-63=0$$

may be written $$9\,(x-1)^2-4(y-3)^2=36;$$

i.e., $$\frac{(x-1)^2}{2^2}-\frac{(y-3)^2}{3^2}=1;$$

hence [Cf. 38 a] its locus is an hyperbola, with center at (1, 3), transverse axis of length 4 and parallel to the x-axis, conjugate axis of length 6; also, $e = \frac{\sqrt{13}}{2}$, its foci the points $(1 - \sqrt{13}, 3)$ and $(1 + \sqrt{13}, 3)$, its directrices the lines $x = 1 - \frac{4}{\sqrt{13}}$ and $x = 1 + \frac{4}{\sqrt{13}}$.

EXERCISES

Determine for each of the following hyperbolas the center, semi-axes, foci, vertices, and latus rectum:

1. $16y^2 - 8x^2 + 64y - 36x + 10 = 0$.
2. $x^2 - 5y^2 + 15y - 10x + 1 = 0$.
3. $2y + 6x + 3x^2 = y^2 + 7$.
4. $Ax^2 - By^2 + 2Gx + 2Fy + C = 0$.
5. Reduce the equations of exercises 1, 2, 3, to the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Sketch each curve.

95. Summary. In the preceding articles it has been shown that the special equation of the second degree,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \tag{1}$$

always represents a conic section, *whose axes are parallel to the coördinate axes.* There are three cases, corresponding to the three species of conic.

(1) *The parabola*: either A or B is zero. In exceptional cases this curve degenerates into a pair of real or imaginary parallel straight lines, and these may coincide. [Arts. 84, 53]

(2) *The ellipse:* neither A nor B is zero, and they have like signs. In exceptional cases this curve degenerates into a circle, a point, or an imaginary locus. [Arts. 89, 86, 53].

(3) *The hyperbola*: neither A nor B is zero, and they have

unlike signs. In exceptional cases this curve degenerates into a pair of real intersecting lines. [Arts. 94, 53].

The ellipse and hyperbola have centers, and therefore are called **central conics**, while the parabola is said to be **non-central.***

The equation for each conic has two standard forms, which state a characteristic geometric property of the curve, and to which all other equations representing that species can be reduced. These standard forms are the simplest for studying the curves; but the student must discriminate carefully between general results and those which hold only when the equation is in the standard form.

96. Polar equation of the conic. Based upon the "focus and directrix" definition already given [Art. 78], the polar equation of a conic section is easily derived.

Let $D'D$ (Fig. 73) be the given line (the directrix) and O the given point (the focus); draw ZOR through O and perpendicular to $D'D$, and let O be chosen as the pole and OR as the initial line. Also let $P \equiv (\rho, \theta)$ be any point on the locus, and let e be the eccentricity. Draw MP and OK parallel, and LP and HK perpendicular to $D'D$, and let $OK = l$; then

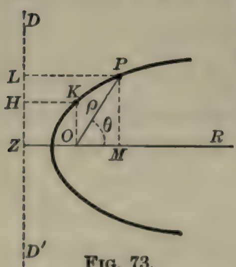

FIG. 73.

$$OP = e \cdot LP, \quad \text{[definition of the curve]}$$

$$= e\,(ZO + OM);$$

$$\therefore \rho = e\left(\frac{l}{e} + \rho\cos\theta\right).$$

This equation, when solved for ρ, may be written in the form

* It is at times more convenient to consider that the latter curve has a center at infinity, on the principal axis.

$$\rho = \frac{l}{1 - e\cos\theta}, \qquad [39]$$

which is the polar equation of a conic section referred to its focus and principal axis; e being the eccentricity and l the semi-latus-rectum. If $e = 1$, equation [39] represents a parabola; if $e < 1$, an ellipse; and if $e > 1$, an hyperbola.

Equation [39] shows that if $e < 1$, *i.e.*, if the equation represents an ellipse, there is no value of θ for which ρ becomes infinite. Therefore there is *no* direction in which a line may be drawn from the focus to meet an ellipse at an infinite distance. If $e = 1$, *i.e.*, if the equation represents a parabola, there is one value of θ, *viz.*, $\theta = 0$, for which ρ becomes infinite. Therefore there is *one* direction in which a line may be drawn to meet a parabola at infinity. If $e > 1$, *i.e.*, if the equation represents an hyperbola, there are two values of θ, *viz.*, $\theta = \pm\cos^{-1}(1 : e)$, for which ρ becomes infinite. Therefore there are *two* directions in which a line may be drawn to meet an hyperbola at infinity.

IV. THE GENERAL EQUATION

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$

Reduction to the Standard Form

97. Condition that the general quadratic expression may be factored: the discriminant. The most general equation of the second degree in two variables may be written in the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0. \qquad (1)$$

If factorable, this equation represents two straight lines. [Arts. 37 and 53]. It is required to find the relation that must exist among the coefficients of this equation in order that its first member may be separated into two rational factors, each

of the first degree, *i.e.*, it is required to find the condition that the equation may be written thus:

$$(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) = 0. \tag{2}$$

Evidently if equation (1) can be written in the form of equation (2), then the values of x obtained from equation (1) are either

$$x = \frac{-c_1 - b_1 y}{a_1} \quad \text{or} \quad x = \frac{-c_2 - b_2 y}{a_2},$$

and are therefore *rational.*

Solving equation (1) for x in terms of y, by completing the square of the x-terms;

$$A^2x^2 + 2A(Hy + G)x + (Hy + G)^2$$
$$= -ABy^2 - 2AFy - AC + (Hy + G)^2, \tag{3}$$

i.e. $Ax + Hy + G$

$$= \sqrt{(H^2 - AB)y^2 + 2(HG - AF)y + G^2 - AC}; \tag{4}$$

and finally,

$$x = -\frac{H}{A}y - \frac{G}{A} \pm \frac{1}{A}\sqrt{(H^2 - AB)y^2 + 2(HG - AF)y + G^2 - AC}. \tag{5}$$

But since x is, by hypothesis, expressible rationally in terms of y, therefore the expression under the radical sign is a perfect square; and therefore

$$(HG - AF)^2 - (H^2 - AB)(G^2 - AC) = 0, \tag{6}$$

i.e., $$\boldsymbol{ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.} \quad [\mathbf{40}]$$

If this condition among the coefficients is fulfilled, then equation (1) has for its locus two straight lines.

The expression $ABC + 2FGH - AF^2 - BG^2 - CH^2$ is called the **discriminant** of the quadratic, and is usually represented by the symbol Δ.

NOTE. The analytic work just given fails if $A = 0$. In that case equation (1) may be solved for y instead of solving it for x, and the same condition, *viz.*, $\Delta = 0$, results. If, however, *both* A and B are zero, then the above method fails altogether.*

In all cases, however, $\Delta = 0$ is the necessary and sufficient condition that the given equation can be factored.

To illustrate the use of equation [40] examine the equation of Art. 53: †

$$2x^2 - xy - 3y^2 + 9x + 4y + 7 = 0.$$

Here $A=2$, $B=-3$, $C=7$, $H=-\frac{1}{2}$, $G=\frac{9}{2}$, and $F=2$;

hence $$\Delta = -42 - 9 - 8 + \tfrac{243}{4} - \tfrac{7}{4} = 0;$$

therefore the first member can be factored.

EXERCISES

Prove that the following equations represent pairs of straight lines:

1. $6y^2 - xy - x^2 + 30y + 36 = 0.$
2. $x^2 - 4xy + 3y^2 + 6y - 9 = 0.$
3. $x^2 - 2xy \sec \alpha + y^2 = 0.$
4. For what value of k will the equation

$$x^2 - 3xy + y^2 + 10x - 10y + k = 0$$

represent two straight lines?

SUGGESTION. Place the discriminant (Δ) equal to zero, and thus find $k = 20$.

Find the values of k for which the following equations represent pairs of straight lines. Find also the equation of

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* For consideration of these special cases see Tanner and Allen's "Elementary Course in Analytic Geometry," pp. 112–113.

† Ordinarily, however, it is better to use the *method* of Art. 97 rather than the resulting equation [40] in solving similar problems.

each line, the point of intersection of each pair of lines, and the angle between them.

5. $6x^2+2kxy+12y^2+22x+31y+20=0.$

6. $12x^2+kxy+2y^2+11x-5y+2=0.$

7. $y^2-kxy-5x+5y=0.$

8. Find the conditions that the straight lines represented by the equation $Ax^2+2Bxy+Cy^2=0$ may be real; imaginary; coincident; perpendicular to each other.

9 Show that $7x^2-48xy-7y^2=0$ is the equation of the bisectors of the angles made by the lines $12x^2+7xy-12y^2=0$. Does the first set of lines fulfill the test of exercise 8 for perpendicularity?

98. The general equation of the second degree with $\Delta \neq 0$. It remains to show that the equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0, \tag{1}$$

with $\Delta \neq 0$, represents a conic section. To prove this it is only necessary to show that, by a suitable change of the coördinate axes, equation (1) may be reduced to the form of equation (1), Art. 95; *i. e.*, that the xy-term may be removed.

Choose new axes OX' and OY', making the angle θ with the corresponding given axes; then [Art. 74],

$$x=x'\cos\theta-y'\sin\theta, \text{ and } y=x'\sin\theta+y'\cos\theta. \tag{2}$$

Substituting these values (2) in equation (1), it becomes

$$\begin{aligned}A(x'\cos\theta-y'\sin\theta)^2&+2H(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)\\&+B(x'\sin\theta+y'\cos\theta)^2+2G(x'\cos\theta-y'\sin\theta)\\&+2F(x'\sin\theta+y'\cos\theta)+C=0,\end{aligned} \tag{3}$$

which, being expanded and rearranged, becomes:

$$x'^2(A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta)$$
$$+ x'y'(-2A\sin\theta\cos\theta - 2H\sin^2\theta + 2H\cos^2\theta + 2B\sin\theta\cos\theta)$$
$$+ y'^2(A\sin^2\theta - 2H\sin\theta\cos\theta + B\cos^2\theta)$$
$$+ x'(2G\cos\theta + 2F\sin\theta)$$
$$+ y'(-2G\sin\theta + 2F\cos\theta) + C = 0. \tag{4}$$

This transformed equation (4) will be free from the term containing the product $x'y'$ if θ be so chosen that

$$-2A\sin\theta\cos\theta - 2H\sin^2\theta + 2H\cos^2\theta + 2B\sin\theta\cos\theta = 0,$$

i.e., if $$2H(\cos^2\theta - \sin^2\theta) = (A - B)2\sin\theta\cos\theta,$$

i.e., if $$2H\cdot\cos 2\theta = (A - B)\sin 2\theta,$$

or finally, if $$\tan 2\theta = \frac{2H}{A - B}. \tag{5}$$

Moreover, it is always possible to choose a positive acute angle θ so as to satisfy this last equation whatever may be the numbers represented by A, B, and H.

Having chosen θ so as to satisfy equation (5), and having substituted the resulting values of $\sin\theta$ and $\cos\theta$ in equation (4), that equation reduces to

$$A'x'^2 + B'y'^2 + 2G'x' + 2F'y' + C = 0, \tag{6}$$

(wherein A', B', $\cdots$ represent the new coefficients)

and therefore represents a conic section with its axes parallel to the new coördinate axes. But equation (6) represents the same locus as equation (1); hence in rectangular coördinates, if $\Delta \neq 0$, *every equation of the form*

$$\boldsymbol{Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0}$$

represents a conic section whose axes are inclined at an angle θ to the given coördinate axes, where θ is determined by the equation

$$\boldsymbol{\tan 2\theta = \frac{2H}{A - B}}. \tag*{[41]}$$

It is to be noted that the constant term C has remained unchanged by the transformation given above.

NOTE. In the proof just given it is assumed that the given axes are at right angles. This restriction may, however, be removed; for if they are not at right angles, a transformation may be made to rectangular axes having the same origin [cf. Art. 75, and Ex. 26, p. 133], and the equation will have its form and degree left unchanged; after which the proof already given applies.

EXAMPLE. Given the equation

$$-x^2+4xy-y^2-4\sqrt{2}\,x+2\sqrt{2}\,y-11=0, \qquad (1)$$

to determine the nature and position of its locus.

Turn the axes through an angle θ, *i.e.*, with

$\tan 2\theta=\frac{4}{0}=\infty$, so that $2\theta=90°$, and $\theta=45°$.

Now, substituting $x=\dfrac{x'-y'}{\sqrt{2}}$ and $y=\dfrac{x'+y'}{\sqrt{2}}$, (2)

equation (1) becomes

$$x'^2-3y'^2-2x'+6y'-11=0, \qquad (3)$$

which represents the same locus as is represented by equation (1).

Equation (2) may be written in the form

$$(x'-1)^2-3(y'-1)^2=9,$$

i.e.,
$$\frac{(x'-1)^2}{3^2}-\frac{(y'-1)^2}{(\sqrt{3})^2}=1, \qquad (4)$$

which represents an hyperbola. [Cf. Art. 93.] Its center is at the point (1, 1); the transverse axis is parallel to the x'-axis;

with $a=3,\quad b=\sqrt{3},$

and $e=\frac{2}{3}\sqrt{3};$

Also, $F\equiv(1+2\sqrt{3},\,1)$

$F'\equiv(1-2\sqrt{3},\,1)$

and the directrices have the equations

$$x' = 1 + \tfrac{3}{2}\sqrt{3}$$

and $$x' = 1 - \tfrac{3}{2}\sqrt{3},$$

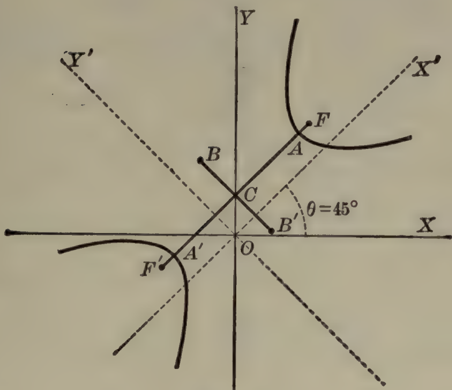

Fig. 74.

respectively; and the latus rectum is 2. These results refer to the new axes, of course, and the locus is that represented in Fig. 74

Example 2. Given the equation

$$4\,x^2 + 4\,xy + y^2 - 18\,x + 26\,y + 64 = 0, \tag{5}$$

to determine the nature and position of its locus. Here $\tan 2\,\theta = \frac{4}{3}$, from which it follows, Art. 12, that

$$\sin 2\,\theta = \tfrac{4}{5} \text{ and } \cos 2\,\theta = \tfrac{3}{5};$$

thence, because

$$\cos 2\,\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$

it is easily deduced that $\sin\theta = \dfrac{1}{\sqrt{5}}$ and $\cos\theta = \dfrac{2}{\sqrt{5}}$.

Then $$x = \frac{2\,x' - y'}{\sqrt{5}}, \text{ and } y = \frac{x' + 2\,y'}{\sqrt{5}}$$

Substituting these values in equation (5), it becomes

$$5x'^2 - 2\sqrt{5}\,x' + 14\sqrt{5}\,y' + 64 = 0,$$

i.e.,
$$\left(x' - \frac{1}{\sqrt{5}}\right)^2 = -\frac{14}{\sqrt{5}}\left(y' + \frac{63}{14\sqrt{5}}\right); \tag{6}$$

this is the equation of a parabola whose vertex is the point $A \equiv \left(\frac{1}{\sqrt{5}},\ -\frac{9}{2\sqrt{5}}\right)$; and whose focus is the point $F \equiv \left(\frac{1}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$. The axis is parallel to the negative end of the y'-axis, and the latus rectum is $\frac{14}{\sqrt{5}}$. These results refer to new axes; the locus of the above equation referred to the old axes is indicated in Art. 79.

EXERCISES

1. For the hyperbola in Fig. 74 find the coördinates of the center and of the foci, and also the equations of its axes and directrices, all referred to the axes OX and OY.

Hint: Make use of the equations of transformation (2) of Art. 98.

By first removing the xy-term, determine the nature and position of the loci represented by the following equations. Also plot the curves.

2. $x^2 - 2\sqrt{3}\,xy + 3y^2 - 24x - 8\sqrt{3}\,y + 16\sqrt{3} = 0.$

3. $x^2 - 4\sqrt{5}\,xy + 2y^2 + \sqrt{5}\,x + 10y = 0.$

4. $2x^2 + 8xy + 8y^2 + x + y + 3 = 0.$

5. $x^2 + 2xy - y^2 + 8x + 4y - 8 = 0.$

Note. The next five Articles, 99–103, may be omitted for a briefer course.

99.* Center of a conic section. As already defined [Arts. 87, 92, 95], the center of a curve is a point such that all chords of the curve passing through it are bisected by it. It has also been shown that such a point exists for the ellipse and hyperbola, *i.e.*, that these are **central conics.**

If the equation of the conic is given in the form

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

the necessary and sufficient condition that the origin is at the center, is $G = 0$ and $F = 0$.

For if the origin is at the center, and (x_1, y_1) is any given point on the locus of equation (1), then $(^-x_1, {}^-y_1)$ must also be on this locus (because these two points are on a straight line through the origin and equidistant from it); hence

$$Ax_1^2 + 2\,Hx_1y_1 + By_1^2 + 2\,Gx_1 + 2\,Fy_1 + C = 0, \tag{2}$$

and $$Ax_1^2 + 2\,Hx_1y_1 + By_1^2 - 2\,Gx_1 - 2\,Fy_1 + C = 0. \tag{3}$$

By subtraction

$$4\,Gx_1 + 4\,Fy_1 = 0;$$

i.e., $$Gx_1 + Fy_1 = 0. \tag{4}$$

But equation (4) is to be satisfied by the coördinates x_1 and y_1 of *every* point on the locus of equation (1), and the necessary and sufficient conditions for this are

$$G = 0 \text{ and } F = 0.$$

100.* Transformation of the equation of a central conic to parallel axes through its center. Let the equation of the given conic be

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

and let the coördinates of its center be α and β. Transform equation (1) to parallel axes through the point (α, β): then $x = x' + \alpha$ and $y = y' + \beta$. This substitution gives

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* See note, p. 168.

$$A(x'+\alpha)^2+2H(x'+\alpha)(y'+\beta)+B(y'+\beta)^2$$
$$+2G(x'+\alpha)+2F(y'+\beta)+C=0;$$

i.e., $Ax'^2+2Hx'y'+By'^2+2x'(A\alpha+H\beta+G)$

$+2y'(H\alpha+B\beta+F)$

$$+A\alpha^2+2H\alpha\beta+B\beta^2+2G\alpha+2F\beta+C=0. \qquad (2)$$

Now, by Art. 99,

$$A\alpha+H\beta+G=0 \text{ and } H\alpha+B\beta+F=0; \qquad (3)$$

solving these equations gives

$$\alpha=\frac{BG-FH}{H^2-AB} \text{ and } \beta=\frac{AF-GH}{H^2-AB}, \qquad [42]$$

which are the coördinates of the center of the locus of equation (1).

It is to be noted here that the new absolute term, *i.e.*, the term free from x' and y' in equation (2), may be obtained by substituting α and β for x and y in the first member of equation (1); also that the terms of second degree are unchanged by the transformation of this article.

101.* Test for the species of a conic. In Art. 100 the center (α, β) is a finite point unless $H^2-AB=0$; therefore, *if* $H^2-AB=0$, *the conic is a parabola.*

That is, for a parabola the terms of second degree, *viz.*, $Ax^2+2Hxy+By^2$, form a perfect square.

In the transformation of Art. 98 the new coefficients found were

$$A'=A\cos^2\theta+2H\sin\theta\cos\theta+B\sin^2\theta; \qquad (1)$$
$$B'=A\sin^2\theta-2H\sin\theta\cos\theta+B\cos^2\theta; \qquad (2)$$
$$2H'=2H\cos 2\theta-(A-B)\sin 2\theta. \qquad (3)$$

Then, by addition of (1) and (2),

$$A'+B'=A+B. \qquad [43]$$

* See note, p. 168.

Again, $A' - B' = 2H\sin 2\theta + (A - B)\cos 2\theta$;

hence, $(A' - B')^2 + 4H'^2 = (A - B)^2 + 4H^2$, (4)

which, combined with equation [43], may be reduced to

$$\boldsymbol{H'^2 - A'B' = H^2 - AB}. \qquad [44]$$

Since the functions $A + B$ and $H^2 - AB$ remain unchanged in value by the transformation of Art. 98, they are called **invariants** of the given equation for that transformation.

Now, since $H' = 0$ when θ is chosen by formula [41], in the transformation of Art. 98, equation [44] gives

$$-A'B' = H^2 - AB;$$

hence, *if* $H^2 - AB > 0$, A' *and* B' *have opposite signs, and the curve is an hyperbola;*

If $H^2 - AB < 0$, A' *and* B' *have like signs, and the curve is an ellipse.*

102.* **Reduction of the given equation to the standard equation of the conic.** By means of the invariants of Art. 101, shorter methods than those of Art. 98 may be used for reducing equations of the second degree to their standard forms:

Case I. *Central Conics.* $\boldsymbol{\Delta \neq 0}$, **and** $\boldsymbol{H^2 - AB \neq 0}$.

(1) Find α and β, by equation [42].

(2) Find C' by the relation

$$C' = G\alpha + F\beta + C;\dagger$$

i.e., by Eq. 42, $\quad C' = -\dfrac{\Delta}{H^2 - AB}.$

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* See note, p. 168.

† For $C' = A\alpha^2 + 2H\alpha\beta + B\beta^2 + 2G\alpha + 2F\beta + C$
$= \alpha(A\alpha + H\beta + G) + \beta(H\alpha + B\beta + F) + G\alpha + F\beta + C$
$= G\alpha + F\beta + C$ by equations (3), Art. 100.

(3) Calculate A' and B' from the two invariants,

$$A' + B' = A + B, \quad -A'B' = H^2 - AB,$$

taking $A' - B'$ with the same sign as H.*

(4) The transformed equation then is

$$A'x^2 + B'y^2 + C' = 0.$$

(5) To plot, construct 2θ from equation [41].

CASE II. *Non-central conic.* **$\Delta \neq 0$, and $H^2 - AB = 0$.**

(1) Factor $Ax^2 + 2Hxy + By^2 \equiv (ax + by)^2$.

(2) With $\tan\theta = -\dfrac{a}{b}$, rotate the axes through the angle θ, [Art. 98]: either A' or B' will vanish with H'.

(3) Change the resulting equation to form of [33, p. 140] or [34, p. 141].

103.† The equation of a conic through given points. The general equation of a conic may be written

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \qquad (1)$$

and contains five parameters, the five ratios between the coefficients A, H, B, G, F, C. Since five equations, or conditions, will determine those parameters, in general five points will determine a conic. That is, in general, *a conic may be made to pass through five, and only five, given points.*

If, however, the conic is to be a parabola, one equation is given; *viz.*, $H^2 - AB = 0$, hence only four additional conditions

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* From Art. 101, equation (4), since $H' = 0$, $A' - B' = \sqrt{(A-B)^2 + 4H^2}$; from equation [41], $\sin 2\theta = \dfrac{2H}{\sqrt{(A-B)^2+4H^2}} = \dfrac{2H}{A'-B'}$.

But θ is chosen in the first quadrant, hence $\sin 2\theta$ is $+$; then H and $A' - B'$ have the same sign.

† See note, p. 168.

are needed or are possible. In general, *a parabola may be made to pass through four points, only.*

A circle has two conditions given, *viz.*, $A = B$, $H = 0$; therefore, in general, *a circle may be made to pass through three points, only.*

A pair of straight lines has one condition given, $\Delta = 0$, therefore, in general, *a pair of straight lines may be made to pass through four points, only.*

The method to be followed in obtaining the equation of the required conic has been used in Art. 57, and may be indicated for finding the equation of the parabola through four given points,

$$P_1 \equiv (x_1, y_1),\ P_2 \equiv (x_2, y_2),\ P_3 \equiv (x_3, y_3), \text{ and } P_4 \equiv (x_4, y_4).$$

The equation must be of the form (1), therefore,

$$Ax_1^2 + 2\,Hx_1y_1 + By_1^2 + 2\,Gx_1 + 2\,Fy_1 + C = 0,$$
$$Ax_2^2 + 2\,Hx_2y_2 + By_2^2 + 2\,Gx_2 + 2\,Fy_2 + C = 0,$$
$$Ax_3^2 + 2\,Hx_3y_3 + By_3^2 + 2\,Gx_3 + 2\,Fy_3 + C = 0,$$
$$Ax_4^2 + 2\,Hx_4y_4 + By_4^2 + 2\,Gx_4 + 2\,Fy_4 + C = 0;$$

also, $$H^2 - AB = 0.$$

The required ratios between the coefficients of equation (1) may be found from these equations.

EXAMPLES ON CONICS

Find the center (or the vertex), the axes, and the species, of the following conics:

1. $x^2 + 5\,xy + y^2 + 8\,x - 20\,y + 15 = 0;$
2. $(x - y)^2 + 2\,x - y = 1;$
3. $3\,x^2 + 2\,y^2 - 2\,x + y - 1 = 0;$
4. $8\,xy - 3\,y^2 + 17\,y + 3\,x^2 - x - 10 = 0;$

5. $9x^2 - 6xy + 9x + y^2 - 3y + 2 = 0$;

6. $3x^2 + xy + 3y^2 = 0$;

7. $3x^2 + 3y^2 + 11x - 5y + 7 = 0$;

8. $x^2 + 2xy - y^2 + 8x + 4y - 8 = 0$;

9. $y^2 + xy - 2x^2 + 6x + 3y = 0$;

10. $9x^2 + 16y^2 + 24xy - 60x - 80y + 100 = 0$.

Trace the following conics:

11. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0$;

12. $4x^2 + 9y^2 - 12x - 24y - 11 = 0$;

13. $3x^2 - 3y^2 + 8xy - 10y + 6x + 5 = 0$;

14. $(x + y)(x + y + 4) + 4 = 0$.

15. What conic is determined by the points (2, 3), (0, ⁻3), (2, 0), (5, 5), and (⁻5, ⁻5)?

16. Find the equation of the parabola through the points (4, 3), (0, ⁻4), (6, 1), (⁻6, 2).

17. Find the equation of the conic through the points (5, 3), (4, 4), (2, 6), (7, ⁻1), (0, 0).

V. SECANTS, TANGENTS, NORMALS, DIAMETERS

104. Tangent to a conic in terms of its point of contact. The equation of a tangent to a conic may be found by the method of Art. 61. That Article should now be reread with care.

105. Tangent to a curve at a given point: second method. A second form of the method of Art. 61 for finding a tangent to a curve is given by an example.

Given two points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_1 + h, y_1 + k)$ on the conic

$$Ax^2 + 2Hxy + By^2 + C = 0. \tag{1}$$

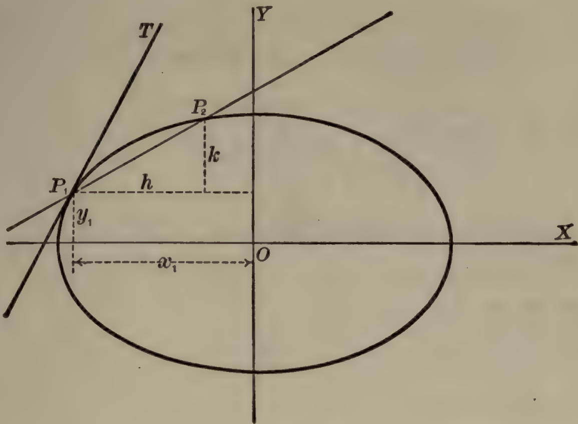

FIG. 75.

Now, the slope of the secant is $\frac{k}{h}$; and as the secant moves so that P_2 approaches P_1 along the curve, this slope approaches as a limit the slope of the tangent P_1T; while h and k approach 0 as a limit;

i.e., $$m_t = \text{limit}\left[\frac{k}{h}\right] \text{ when } h \doteq 0, \text{ and } k \doteq 0.$$

Now, from Eq. (1),

$$Ax_1^2 + 2\,Hx_1y_1 + By_1^2 + C = 0. \tag{2}$$

$$A(x_1+h)^2 + 2\,H(x_1+h)(y_1+k) + B(y_1+k)^2 + C = 0; \tag{3}$$

i.e., $$Ax_1^2 + 2\,Hx_1y_1 + By_1^2 + 2\,hAx_1 + 2\,kBy_1 + 2\,kHx_1 + 2\,hHy_1 + Ah^2 + 2\,Hhk + Bk^2 + C = 0. \tag{4}$$

Subtracting (2) from (4), and factoring,

$$h(2\,Ax_1 + 2\,Hy_1 + Ah + Hk) + k(2\,Hx_1 + 2\,By_1 + Hh + Bk) = 0. \tag{5}$$

Hence $$\frac{k}{h} = -\frac{2\,Ax_1 + 2\,Hy_1 + Ah + Hk}{2\,Hx_1 + 2\,By_1 + Hh + Bk}. \tag{6}$$

Then $$\lim_{h \doteq 0,\ k \doteq 0} \left[\frac{k}{h}\right] = -\frac{2\,Ax_1 + 2\,Hy_1}{2\,Hx_1 + 2\,By_1} = -\frac{Ax_1 + Hy_1}{Hx_1 + By_1}. \qquad (7)$$

Now the equation of the tangent becomes

$$y - y_1 = -\frac{Ax_1 + Hy_1}{Hx_1 + By_1}(x - x_1). \qquad (8)$$

Multiplying and rearranging, equation (8) becomes

$$(Ax_1 + Hy_1)\,x + (Hx_1 + By_1)\,y - (Ax_1^2 + 2\,Hx_1y_1 + By_1^2) = 0; \quad (9)$$

which, by (2), may be written

$$Ax_1x + H(xy_1 + x_1y) + By_1y + C = 0, \qquad (10)$$

the required equation.

EXERCISES

Derive in full, by the secant method, the following equations of tangents, the point of contact being $P_1 \equiv (x_1, y_1)$:

1. For the parabola $y^2 = 4\,px$; $\mathbf{y_1y = 2\,p(x + x_1)}$. [45]

2. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $\mathbf{\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1}$. [46]

3. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $\mathbf{\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1}$. [47]

4. For the hyperbola $2\,xy = c$; $\mathbf{xy_1 + x_1y = c}$. [48]

5. For the conic $Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$;

$$Ax_1x + H(y_1x + x_1y) + By_1y + G(x + x_1) + F(y + y_1) + C = 0.$$

Notice that the equation of the tangent for each conic through a given point of contact (x_1, y_1) may be written as follows: *in the equation of the conic substitute x_1x for x^2, y_1y for y^2, $\frac{xy_1 + x_1y}{2}$ for xy, $\frac{x + x_1}{2}$ for x, and $\frac{y + y_1}{2}$ for y.*

106. Normal to a conic. The equation of a normal to a conic can be obtained at once from that of the tangent, by the method of Art. 62.

EXAMPLE. To find the equation of the normal to the ellipse $\frac{x^2}{18}+\frac{y^2}{8}=1$ at the point (3, 2).

The equation of the tangent at the point (3, 2) is

$$\frac{3x}{18}+\frac{2y}{8}=1; \quad i.e., \; 2x+3y=12.$$

The perpendicular line through (3, 2) is $3x-2y=5$, which is, therefore, the required normal.

EXERCISES

1. Is the line $4x+9y=35$ tangent to the ellipse $2x^2+3y^2=35$?

2. Find the equation of a tangent to the conic $x^2+5y^2-3x+10y-4=0$, parallel to the line $y=4x+10$. [Art. 59].

Write the equations of the tangent and normal to each of the following conics, through a point (x_1, y_1) on the curve:

3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ 4. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

5. $x^2=4p(y+5)$; sketch the figure.

6. $4x^2-5y^2-12x=0$; sketch the figure.

7. $x^2+3y^2-3x+12y+9=0$; sketch the figure.

8. Derive, by the secant method, the tangent to the ellipse $4x^2+y^2+8x-2y+1=0$, the point of contact being (x_1, y_1).

Write the equations of the tangents and normals to each of the following conics, at the given points; also sketch each figure:

9. $9x^2+5y^2+36x+20y+11=0$, at the point $(^-2, 1)$;

10. $9x^2+4y^2+6x+4y=0$, at the positive end of the latus rectum;

11. $y^2-6y-8x=31$, at the negative end of the latus rectum;

12. $\frac{x^2}{5}+\frac{y^2}{5}=1$, at the point $(\sqrt{4}, 1)$;

13. $4x^2+3y^2=16$, at the point $(1, {}^-2)$.

14. Find the equation of the tangent to the parabola $y^2-6y-8x-31=0$ passing through the point $({}^-4, {}^-1)$ *not* on the curve.

107. The equation of the tangent to a conic in terms of its slope. The equation of a line having the given slope m is

$$y=mx+k; \tag{1}$$

it is desired to find that value of k for which this line will become tangent to a given conic, *e.g.*, to the parabola whose equation is

$$y^2=4px. \tag{2}$$

Considering equations (1) and (2) as simultaneous, and eliminating y, gives the resulting equation,

$$(mx+k)^2=4px, \tag{3}$$

which has for its roots the abscissas of the two points in which the loci of equations (1) and (2) intersect. These roots will be equal, and therefore the points of intersection will be coincident, if

$$(mk-2p)^2-m^2k^2=0,$$

i.e., if

$$k=\frac{p}{m}. \tag{4}$$

Therefore,

$$\boldsymbol{y=mx+\frac{p}{m}} \qquad [\mathbf{49}]$$

is, for all values of m, the equation of the required tangent.

Similarly, the tangent with slope m

for the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx\pm\sqrt{a^2m^2+b^2}$; [50]

for the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=mx\pm\sqrt{a^2m^2-b^2}$. [51]

108. Diameter of a conic section. The locus of the middle points of any system of parallel chords of a given conic is called a **diameter** of that conic, and the chords which that diameter bisects are called the **chords of that diameter.** This definition leads at once to the equation of a diameter, as will be shown for the ellipse.

Let m be the slope of the given system of parallel chords of the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

and let
$$y=mx+c \tag{2}$$

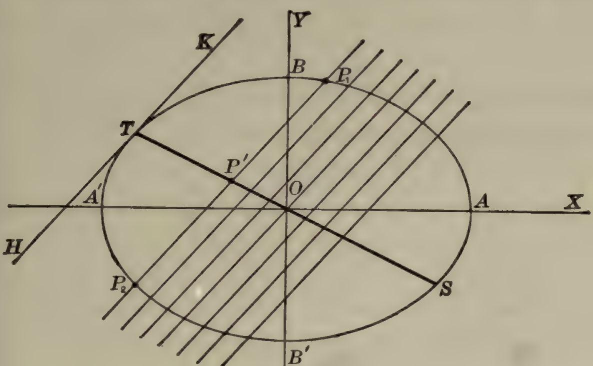

FIG. 76.

be the equation of one of these chords, which meets the curve in the two points $P_1\equiv(x_1, y_1)$ and $P_2\equiv(x_2, y_2)$. Let $P\equiv(x', y')$ be the middle point of this chord, so that

$$x' = \frac{x_1 + x_2}{2}, \quad y' = \frac{y_1 + y_2}{2}. \tag{3}$$

The coördinates of P_1 and P_2 are found by solving (1) and (2) as simultaneous equations, therefore the abscissas x_1 and x_2 are the roots of the equation

$$(a^2m^2 + b^2)x^2 + 2\,a^2cmx + a^2c^2 - a^2b^2 = 0; \tag{4}$$

hence, by Art. 8,
$$x' = -\frac{a^2cm}{a^2m^2 + b^2}, \tag{5}$$

Then, by Eq. (2)
$$y' = \frac{b^2c}{a^2m^2 + b^2}. \tag{6}$$

Now, by varying the value of c, equations (5) and (6) give the coördinates of the middle point for each of the chords of the given set. It is required to find the locus of P' for all values of c, *i.e.*, to find an equation satisfied by x' and y' and not dependent upon the value of c. If x' is divided by y', then c is eliminated from the equations (5) and (6), giving

$$\frac{x'}{y'} = -\frac{a^2}{b^2}m. \tag{7}$$

Therefore the coördinates of the middle point of every chord of slope m satisfy the equation

$$\frac{x}{y} = -\frac{a^2}{b^2}m,$$

i. e.,
$$y = -\frac{b^2}{a^2m}x; \tag{52}$$

which is therefore the equation of the required diameter.

Likewise it may be shown that the equation of the diameter

of the hyperbola is
$$y = \frac{b^2}{a^2m}x, \tag{53}$$

of the parabola is
$$y = \frac{2p}{m}; \tag{54}$$

using tne first standard equation of each conic.

109. Equation of a conic that passes through the intersections of two given conics. Let the given conics be

$$S_1 \equiv A_1x^2 + B_1y^2 + 2\,G_1x + 2\,F_1y + C_1 = 0, \tag{1}$$

and

$$S_2 \equiv A_2x^2 + B_2y^2 + 2\,G_2x + 2\,F_2y + C_2 = 0; \tag{2}$$

then, if k be any constant whatever,

$$S_1 + kS_2 = 0 \tag{3}$$

represents a conic whose axes are parallel to the coördinate axes [Art. 95], and which passes through the points in which the conics $S_1 = 0$ and $S_2 = 0$ intersect each other [Art. 38]; *i.e.*, $S_1 + kS_2 = 0$ represents a *family* of conics, each member of which passes through the intersections of $S_1 = 0$ and $S_2 = 0$. The parameter k may be so chosen that the conic (3) shall, in addition to passing through the four points in which $S_1 = 0$ and $S_2 = 0$ intersect, satisfy one other condition; *e.g.*, that it shall pass through a given fifth point.

Moreover, if $S_1 = 0$ and $S_2 = 0$ are both circles, then $S_1 + kS_2 = 0$ is also a circle. [Cf. Arts. 65 and 66].

EXAMPLES ON CHAPTER VIII

1. Find the equations of those tangents to the conic $4\,x^2 - 5\,y^2 = 20$, which pass through the point $(0, {}^-2)$. [Use equation $y + 2 = mx$].

2. Find a conic through the intersections of the ellipse $4\,x^2 + y^2 = 16$ and the parabola $y^2 = 4\,x + 4$, and also passing through the point 0, 0. What species of conic is it?

3. Show that the curves $4\,x^2 + 9\,y^2 = 36$ and $2\,x^2 - 3\,y^2 = 6$ have the same foci, and that they cut each other at right angles.

4. Find the vertices of an equilateral triangle circumscribed about the ellipse $9\,x^2 + 16\,y^2 = 144$, one side being parallel to the major axis of the curve.

5. Find the normal to the conic $3x^2 + y^2 - 2x - y = \frac{5}{12}$, making the angle $\tan^{-1}(-\frac{1}{3})$ with the x-axis.

6. Show that the equation of the secant line that cuts the ellipse $4x^2 + 9y^2 - 36 = 0$ in the points $P_1 \equiv (x_1, y_1)$ and $P_2 = (x_2, y_2)$ may be written

$$4(x - x_1)(x - x_2) + 9(y - y_1)(y - y_2) = 4x^2 + 9y^2 - 36.$$

Determine the species of the following conics; and also their foci, directrices, centers, semi-axes, and latera recta:

7. $y^2 = (x - 3)(x + 4)$;

8. $3x^2 - 2y^2 - 6x - 4y = 7$;

9. $x^2 + a^2 = 4ax + 4py + 4pb$;

10. $2x^2 + 3y^2 + 16x + 12y + 38 = 0$;

11. $9y^2 - 4x^2 = 36y$;

12. $x^2 + 2y^2 - 2 = a(2x - 4y + 9a)$.

13. Show that the polar equation of the parabola, with its vertex at the pole, is $\rho = \dfrac{4p\cos\theta}{\sin^2\theta}$.

14. Show that if the left-hand focus be taken as pole, the polar equation of the ellipse is $\rho = \dfrac{a(1 - e^2)}{1 - e\cos\theta}$.

15. Derive the polar equation of an hyperbola, with its pole at the focus, eccentricity 3, and the distance of the focus from the directrix equal to 9.

16. From equation [39], trace the parabola.

17. From equation [39], trace the ellipse.

18. By means of equation [39], prove that the length of a chord through the focus of a parabola, and making an angle of 30° with the axis of the curve, is four times the length of the latus rectum.

CHAPTER IX

SPECIAL PROPERTIES OF THE CONICS

110. In the present chapter, some of the intrinsic properties of the conic are to be studied, *i.e.*, properties which belong to the curves and are entirely independent of the position of the coördinate axes. For this purpose, it will, in general, be easier to use the simplest form of the equation of the curve in each case.

THE PARABOLA $y^2 = 4px$

111. Recapitulation. The following data may be assumed for the further study of the parabola:

The eccentricity is	$e = 1;$
the standard equation	$y^2 = 4px;$
the focus is the point	$F \equiv (p, 0);$
the directrix is the line	$x = -p;$
the axis of the curve is	$y = 0;$
the equation of the tangent at (x_1, y_1),	$y_1 y = 2p(x + x_1);$
or, in terms of its slope,	$y = mx + \dfrac{p}{m};$
the equation of the diameter	$y = \dfrac{2p}{m},$

where m is the slope of the chords bisected.

112. Construction of the parabola. The two conceptions of a locus given in Art. 28 lead to two methods for constructing

a curve: (1) by plotting points to be connected by a smooth curve, and (2) by the motion of a point constrained by some mechanical device to satisfy the law which defines the curve. These two methods may be used in constructing a parabola.

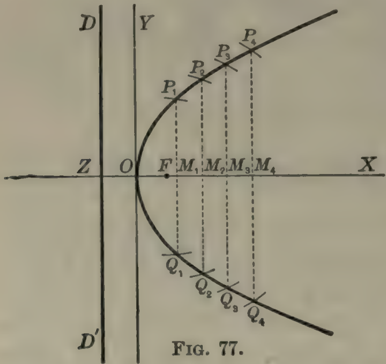

FIG. 77.

(1) *By separate points.* Given the focus F and the vertex O, draw the axis OFX, the directrix $D'D$ cutting this axis in Z, and also a series of lines perpendicular to the axis at M_1, M_2, M_3, etc., respectively. With F as center and ZM_1 as radius, describe arcs cutting the line at M_1 in two points P_1 and Q_1; similarly, with F as center and ZM_2 as radius, cut the line at M_2 in P_2 and Q_2; and so on. The points thus found evidently satisfy the definition of the parabola [Art. 79]. If these are then connected by a smooth curve, it will be approximately the required parabola.

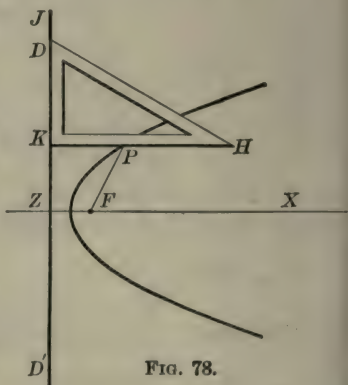

FIG. 78.

(2) *By a continuously moving point.* Let $D'D$ be the directrix and F the focus. Place a right triangle with its longer side KH in coincidence with the axis of the curve, and its shorter side KJ in coincidence with the directrix. Let one end of a string of length KH be fastened at H, and the other end at F. If now a pencil

point be pressed against the string, keeping it taut while the triangle is moved along the directrix, as indicated in the figure, then, in every position of P,

$$FP = KP,$$

therefore the pencil will trace an arc of a parabola.

113. Subtangent and subnormal. Construction of tangent and normal. Let

$$P_1 \equiv (x_1, y_1)$$

be any given point on the parabola

$$y^2 = 4px. \qquad (1)$$

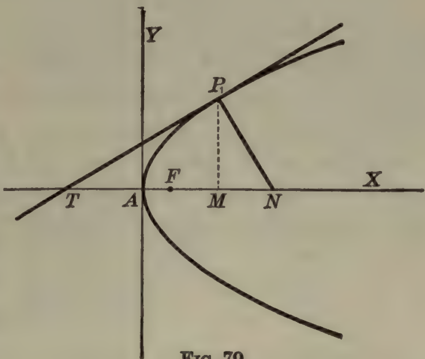

FIG. 79.

Draw the ordinate MP_1, the tangent TP_1, and the normal P_1N. Then [Art. 63], the subtangent is TM, the subnormal is NM, the tangent length TP_1, and the normal length NP_1. The tangent at P_1 has for its equation

$$y_1y = 2p(x + x_1), \qquad (2)$$

hence its x-intercept is $-x_1$, $[=AT]$. But $AM = x_1$,

therefore $$TM = 2x_1.$$

This proves that *the subtangent of the parabola $y^2 = 4px$ is bisected at the vertex;* and that *its length is equal to twice the abscissa of the point of contact.*

The normal at P_1 has for its equation [Art. 106]

$$y - y_1 = -\frac{y_1}{2p}(x - x_1), \qquad (3)$$

hence its x-intercept is $x_1 + 2p$, $[=AN]$. But $AM = x_1$,

therefore $$NM = -2p.$$

That is, *the subnormal of the parabola* $y^2 = 4px$ *is constant; it is numerically equal to half the latus rectum.*

These properties suggest simple methods of constructing the tangent and normal to any parabola at a given point P_1, if the axis AX and the focus F of the parabola are given:

First method: from P_1 draw $P_1M \perp AX$, meeting it in M. The vertex being A, construct the point T on AX produced, so that $TA = AM$. The straight line TP_1 is the required tangent at P_1, and a line through P_1 at right angles to this tangent is the required normal.

Second method: from M construct N on AX so that $MN = 2\,AF$ ($\because 2p = 2\,AF$); then P_1N is the required normal, and a line through P_1 at right angles to P_1N is the required tangent.

EXERCISES

1. Construct a parabola with focus 4 cm. from the directrix.

2. Construct a parabola with latus rectum equal to 8.

3. Find the equations of the two tangents to the parabola $y^2 = 4px$, which form with the tangent at the vertex a circumscribed equilateral triangle. Find also the ratio of the area of this triangle to the area of the triangle whose vertices are the points of tangency.

4. Find the equation of a tangent to the parabola $y^2 = 4px$, perpendicular to the line $3y + 4x + 3 = 0$, and find its point of contact.

5. Find the equations of the two tangents to the parabola $y^2 = -5x$ from the point $(2, 2)$, using $y - 2 = m(x - 2)$.

6. Write the equations of the tangents to the parabola $y^2 = -20x$, at the extremities of the latus rectum. On what line do these tangents intersect?

7. Write the equations of the tangent and normal to the parabola $y^2 = 10x$, at the point $(10, {}^-10)$.

8. Write the equation of the normal to the parabola $x^2 = -4y$, drawn through the point $(1, {}^-\frac{1}{4})$.

9. Write the equation of the tangent to the parabola $y^2 = 4px$, (1) for the point for which the normal length equals twice the tangent; (2) for the point for which the normal length is equal to the difference between the subtangent and subnormal.

10. Two equal parabolas have the same vertex, and their axes are at right angles; find the equation of their common tangent, and show that the points of contact are each at the extremity of a latus rectum.

11. Find the locus of the middle point of the normal length of the parabola $y^2 = 4px$.

12. For the point (5, 4) the subtangent of a parabola is 10; construct the curve.

13. Find the subtangent, also the normal length, for the positive end of the latus rectum of the parabola $y^2 = 12x$.

14. Find the equation of a parabola which is tangent to $ax + by = c$, whose vertex is at the origin, and whose axis is parallel to the x-axis.

15. Show that the sum of the subtangent and subnormal for any point on the parabola $y^2 = 4px$, equals one half the length of the focal chord parallel to the corresponding tangent.

16. Show that as the abscissa in the parabola $y^2 = 4px$ increases from 0 to ∞, the absolute value of the slope of the tangent changes from ∞ to 0; hence the curve is concave toward its axis.

17. State for the parabola the properties of the subtangent and subnormal [Art. 113] with relation to the principal axis, instead of to the x-axis.

114. Some properties of the parabola which involve tangents and normals. Let F be the focus, A the vertex, AX the axis,

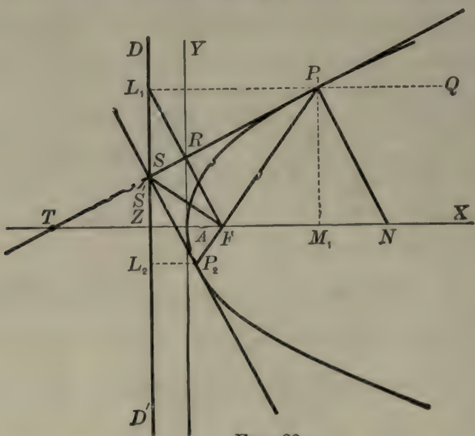

FIG. 80.

and $D'D$ the directrix of the parabola whose equation is

$$y^2 = 4px. \tag{1}$$

Through any point $P_1 \equiv (x_1, y_1)$ on the curve draw the tangent TP_1, cutting the y-axis in R, the directrix in S, and the x-axis in T; also draw the normal P_1N; the focal chord P_2FP_1; the tangent at P_2; the lines L_1P_1Q and L_2P_2, perpendicular to the directrix; and the lines SF and L_1F. Then the following properties of the parabola are readily obtained:

(1) *The focus is equidistant from the points* P_1, T, *and* N.

For $\quad FP_1 = L_1P_1 = ZA + AM_1 \quad = x_1 + p,$

$\quad TF = TA + AF = x_1 + p,$ [Art. 113]

and $\quad FN = AM_1 + (M_1N - AF) = x_1 + p;$ [Art. 113]

hence $\quad FP_1 = TF = FN.$

Or; the point F is the midpoint of the hypotenuse of the right triangle TP_1N, and is therefore equidistant from the vertices T, P_1, and N.

Thus a third method is suggested for constructing the tangent and normal at P_1, *viz.*, by means of a circle, with the focus F as center, and the focal radius FP_1 as radius, which cuts the axis in T and N.

(2) *The tangent and normal bisect internally and externally, respectively, the angle between the focal radius to the point of contact and the perpendicular from that point to the directrix.*

For $\quad \angle L_1P_1T = \angle P_1TF,\quad$ since $L_1P \parallel TF$;

and $\quad \angle TP_1F = \angle P_1TF,\quad$ since $TF = FP_1$;

hence $\quad \angle L_1P_1T = \angle TP_1F.$

Also, $\quad \angle FP_1N = \angle NP_1Q,\quad$ since $P_1N \perp P_1T$.

EXERCISES

1. The focal distance of any point of the parabola $y^2 = 4px$ is $p + x$.

2. Show that $\left(\dfrac{p}{m^2}, \dfrac{2p}{m}\right)$ is the point of contact of the tangent $y = mx + \dfrac{p}{m}$ with the parabola.

3. The equation of the normal to the parabola $y^2 = 4px$, with slope m, is $y = mx - 2pm - pm^3$.

4. Through any point in the plane two tangents may be drawn to the parabola.

5. Through any point in the plane three normals may be drawn to the parabola.

6. The line joining any point in the directrix to the focus of a parabola is a perpendicular to the chord of contact corresponding to that point, [Fig. 80].

7. A perpendicular let fall from the focus upon a tangent meets that tangent upon the tangent at the vertex, [Fig. 80].

8. Write the equations of the normals drawn through the point $(0, {}^{-}3)$ to the parabola $y^2 = -10x$.

9. The circle on a focal chord as diameter is tangent to the directrix, [Fig. 80].

10. The angle between two tangents to a parabola is one half the angle between the focal radii of the points of tangency.

11. The product of the segments of any focal chord of the parabola $y^2 = 4px$ equals p times the length of the chord.

12. Two tangents are drawn from an external point $P_1 \equiv (x_1, y_1)$ to a parabola, and a third is drawn parallel to their chord of contact. The intersection of the third with each of the other two is halfway between P_1 and the corresponding point of contact.

13. The area of a triangle formed by three tangents to a parabola is one half the area of the triangle formed by the three points of tangency.

14. The tangent at any point of the parabola will meet the directrix and latus rectum produced, in two points equidistant from the focus.

15. The normal at one extremity of the latus rectum of a parabola is parallel to the tangent at the other extremity.

16. The tangents at the ends of the latus rectum are twice as far from the focus as they are from the vertex.

17. The circle on any focal radius as diameter touches the tangent drawn at the vertex of the parabola.

115. Some properties of the parabola involving diameters. The equation of the diameter of the parabola [Art. 108],

$$y = \frac{2p}{m}, \tag{1}$$

shows at once that *every diameter of the parabola is parallel to the axis of the curve.*

Conversely, since any value whatever may be assigned to m, each value determining a system of parallel chords, equation (1) may represent *any* line parallel to the x-axis, and therefore *every line parallel to the axis of a parabola bisects some set of parallel chords, and is a diameter of the curve.*

Again, each of the chords cuts the parabola in general in two distinct points, and the nearer these chords are to the extremity of the diameter the nearer are these two points to each other and to their midpoint. In the limiting position, when the chord passes through the extremity of the diameter, the two intersection points and their midpoint become coincident, and the chord is a tangent. Therefore *the tangent at the end of a diameter is parallel to the bisected chords.*

It follows from the preceding properties, or directly from equation (1), that *the axis of the parabola is the only diameter perpendicular to the tangent at its extremity.*

EXERCISES

1. Find the diameter of $y^2 = 16x$ which bisects the chords parallel to the line $3x - 4y + 10 = 0$.

2. A diameter of the parabola $y^2 = -8x$ passes through the point $(2, {}^-3)$; what is the equation of its corresponding chords?

3. Find the equation of the diameter of the parabola $y^2 = -4x + 8$ which bisects the chords $2y - 3x = k$.

4. Find the equation of the tangent to the parabola $(y+6)^2 = -4(x+2)$ which is perpendicular to the diameter $y + 4 = 0$.

5. The tangents at the extremities of a chord of a parabola intersect upon the corresponding diameter.

6. Show how the properties of Art. 115 give a method for constructing a diameter to a set of chords, and in particular for constructing the axis of a given parabola.

7. Show how the problem to construct a tangent and normal to a given parabola at a given point can be solved when the axis is not at first given. [Cf. Art. 113].

REVIEW EXERCISES

Find the equation of a parabola with axis parallel to the x-axis:

1. Passing through the points $(0, 0)$, $(4, 2)$, $(4, {}^-2)$;

2. Passing through the points $(\frac{25}{8}, 0)$, $(4, -1)$, $(\frac{41}{8}, -2)$;

3. Through the point $(0, {}^-5)$, with its vertex at the point $({}^-2, {}^-1)$.

4. A parabola whose axis is parallel to the y-axis passes through the points $({}^+1, {}^-1)$, $({}^-2, {}^-10)$, and $(3, {}^-5)$; find its equation.

5. Find the vertex and axis of the parabola of Ex. 4.

Find the equation of a parabola:

6. If the axis and directrix are taken as coördinate axes.

7. With the focus at the origin, and the y-axis parallel to the directrix.

8. Tangent to the line $3y = -4x - 12$, the equation being in the simplest standard form.

9. If the axis of the parabola coincides with the x-axis, and a focal radius of length 5 coincides with the line $8y - 6x + 3 = 0$.

10. Two equal parabolas have the same vertex, and their axes are perpendicular; find their common chord and common tangent. [Cf. Ex. 10, p. 187].

11. At what angle do the parabolas of Ex. 10 intersect?

12. Two tangents to a parabola are perpendicular to each other; find the product of the corresponding subtangents.

Find the locus of the middle point:

13. Of all the ordinates of a parabola.

14. Of all chords that meet at the foot of the axis, Z [Fig. 80].

15. From any point on the latus rectum of a parabola, perpendiculars are drawn to the tangents at its extremities; show that the line joining the feet of these perpendiculars is a tangent to the parabola.

16. A double ordinate of the parabola $y^2 = 4px$ is $8p$; prove that the lines from the vertex to its two ends are perpendicular to each other.

Two tangents of slope m and m', respectively, are drawn to a parabola; find the locus of their intersection:

17. If $mm' = k$;

18. If $\frac{1}{m} + \frac{1}{m'} = k$;

19. If $\frac{1}{m} - \frac{1}{m'} = k$.

20. Find the locus of the center of a circle which passes through a given point and touches a given line.

21. The latus rectum of the parabola is a third proportional to any abscissa and the corresponding ordinate.

22. Find the locus of the point of intersection of tangents drawn at points whose ordinates are in a constant ratio.

23. What is the equation of the chord of the parabola $y^2 = -4x$ whose middle point is at $(-3, 1)$?

24. Find the locus of the center of a circle which is tangent to a given circle and also to a given straight line.

25. Find the equation to all parabolas which are touched by the straight lines $y = \pm \frac{x}{2}$.

26. At the point P a normal is drawn; find the coördinates of the point where it meets the curve again, and the length of the intercepted chord.

THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

116. Recapitulation. In further study of the ellipse the following data may be assumed:

The standard equation,	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$
the eccentricity,	$e^2 = \frac{a^2 - b^2}{a^2};$
its foci,	$F_1 \equiv (-ae, 0),\ F_2 \equiv (ae, 0);$
its directrices,	$x = -\frac{a}{e},\ x = \frac{a}{e};$
its tangent at P_1,	$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1;$

its tangent with slope m, $$y = mx \pm \sqrt{a^2m^2 + b^2};$$

its diameter, $$y = -\frac{b^2}{a^2m}x,$$

for chords whose slope is m.

Clearly, if the axes are equal, so that $b = a$, the curve takes the special form of the circle, with eccentricity $e = 0$, the two foci coincident at the center, and the directrices infinitely distant.

117. The sum of the focal distances of any point on an ellipse is constant; it is equal to the major axis.

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has its foci at the points

$$F_1 \equiv (-ae, 0) \quad \text{and} \quad F_2 \equiv (ae, 0);$$

with $b^2 = a^2 - a^2e^2$. [Cf. Art. 86].

Let $P_1 \equiv (x_1, y_1)$ be any point on the curve,

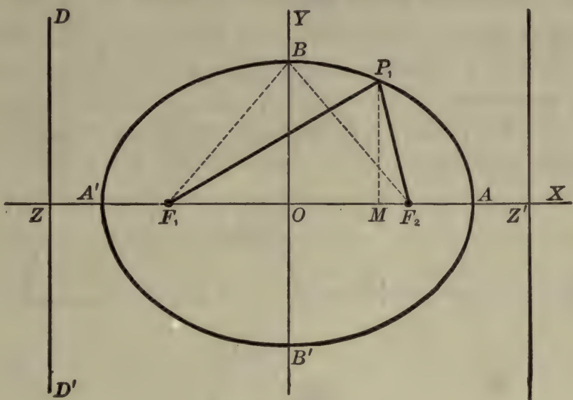

FIG. 81.

then $$F_1P_1 = e(ZO + OM) = e\left(\frac{a}{e} + x_1\right)$$
$$= a + ex_1.$$

Also $$F_2P_1 = e(OZ' - OM)$$
$$= a - ex_1.$$

Hence, by addition, $F_1P_1 + F_2P_1 = 2a$;

i.e., *the sum of the focal distances of any point on an ellipse is constant; it is equal to the major axis.*

This property gives an easy method of finding the foci of an ellipse when the axes $A'A$ and $B'B$ are given.

For $$F_1B + F_2B = 2a;$$
but $$F_1O = OF_2,$$
$$\therefore \quad F_2B = F_1B = a.$$

Hence, to find the foci, describe arcs with B as center and $a = OA$ as radius, cutting $A'A$ in the points F_1 and F_2; these points are the required foci.

118. Construction of the ellipse. The property of Art. 117 is sometimes given as the definition of the ellipse; viz., *the ellipse is the locus of a point the sum of whose distances from two fixed points is constant.* This definition leads at once to the equation of the curve [cf. Ex. 32, p. 56], and also gives a ready method for its construction.

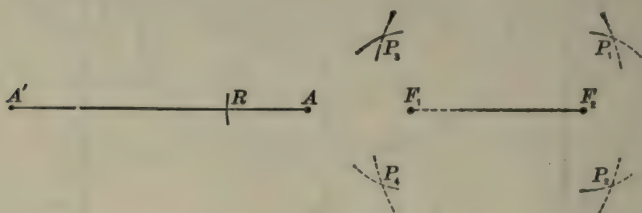

FIG. 82.

(*a*) *Construction by separate points.* Let $A'A$ be the given sum of the focal distances, *i.e.*, the major axis of the ellipse; and F_1 and F_2 be the given fixed points, the foci. With either

focus as center, and with any radius $A'R \gtreqless A'A$ describe an arc: then with the other focus as center, and radius RA, describe an arc cutting the first arc in two points. These are points of the ellipse. In the same way as many points as desired may be constructed; a smooth curve connecting these points is approximately an ellipse.

(β) *Construction by a continuously moving point.* Fix two upright pins at the foci, and over them place a loop of string, equal in length to the major axis plus the distance between the foci. Press a pencil point against the cord so as to keep it taut. As the pencil moves around the foci, it will trace an ellipse.

EXERCISES

1. Construct an ellipse with semi-axes 4 cm. and 2 cm.

2. Construct an ellipse with semi-axes 5 cm. and 10 cm.

3. Construct an ellipse with the distance between the foci 20, and the major axis of length 24.

4. Find the equation of a tangent, and also of a normal, to the ellipse $x^2 + 4y^2 = 16$, each perpendicular to the line $4x + 3y = 5$.

5. By employing equation $y = mx \pm \sqrt{a^2m^2 + b^2}$, find the tangent to the ellipse $16x^2 + 25y^2 = 400$, passing through the point (3, 4).

6. By the method of Ex. 16, p. 188, show that an ellipse is concave toward its center.

7. Through what point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ must a tangent and normal be drawn, to form with the x-axis an isosceles triangle?

8. Write the equations of the tangent and normal at the positive end of the latus rectum of the ellipse $9x^2 + 16y^2 = 144$. Where do these lines cut the x-axis?

9. Tangents to the ellipse $4x^2 + 3y^2 = 5$ are inclined at 45° to the x-axis; find the points of contact.

10. Find the equation of an ellipse (center at the origin), such that the subtangent for the point $(3, \frac{12}{5})$ is $(-\frac{16}{3})$.

11. Find the equations of the tangents from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.

12. Find the tangents to the ellipse $7x^2 + 8y^2 = 56$ which make the angle $\tan^{-1} 3$ with the line $x - y + 1 = 0$.

13. Find the product of the two segments into which a focal chord is divided by the focus of an ellipse, — using Art. 96.

14. Find the points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, such that the tangent makes equal (numerical) angles with the axes; such that the subtangent equals the subnormal.

119. Auxiliary circles. Eccentric angle. The circumscribed and inscribed circles for the ellipse (Fig. 83) are called **auxiliary** circles, and bear an important part in the theory of the ellipse. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

The circle described on its major axis as diameter is called the **major auxiliary circle**; its equation is

$$x^2 + y^2 = a^2; \tag{2}$$

and the circle on the minor axis as diameter is the **minor auxiliary circle**; its equation is

$$x^2 + y^2 = b^2. \tag{3}$$

If $\angle AOQ$ is any angle ϕ at the center of the ellipse, with the initial side on the major axis, and the terminal side cutting the auxiliary circles in R and Q, respectively; and if P

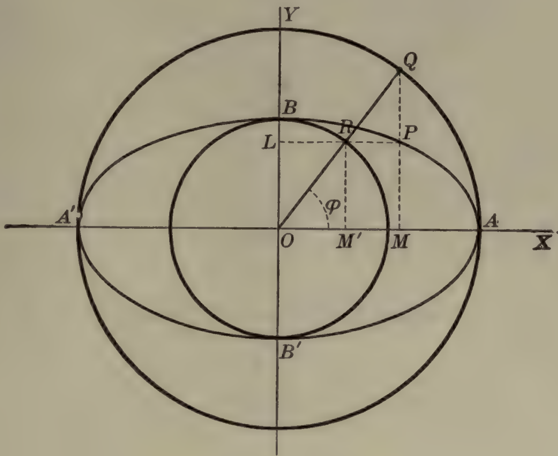

FIG. 83.

is the intersection of the abscissa LR with the ordinate MQ, then P is a point on the ellipse.

For the coördinates of P are

$$OM = OQ \cos \phi \text{ and } MP = M'R = OR \sin \phi,$$

i.e., $$x = a \cos \phi, \qquad y = b \sin \phi. \qquad [55]$$

Now these values satisfy the equation of the ellipse; for, substituting them in equation (1), gives

$$\frac{a^2 \cos^2 \phi}{a^2} + \frac{b^2 \sin^2 \phi}{b^2} = \cos^2 \phi + \sin^2 \phi = 1;$$

hence P is a point of the ellipse.

The points P, Q, and R are called **corresponding points.** The

angle ϕ is the **eccentric angle** of the point P;* and the two equations [55] are equations of the ellipse in terms of the eccentric angle, for together they express the condition that the point P is on the ellipse (1).†

Since, in the figure, $\triangle$ $OM'R$ and OMQ are similar, it follows

that $$MP : MQ = OR : OQ = b : a,$$

and $$OM' : OM = OR : OQ = b : a;$$

that is, *the ordinate of any point on the ellipse is to the ordinate of the corresponding point on the major auxiliary circle in the ratio* $(b : a)$ *of the semi-axes.* Similarly for the abscissas of the corresponding points R and P.

120. The subtangent and subnormal. Construction of tangent and normal.

Let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

be a given ellipse,

then $$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \qquad (2)$$

is the tangent to it at a point $P_1 \equiv (x_1, y_1)$. Let this tangent cut the x-axis at the point T. Draw the ordinate MP_1.

Then the subtangent is, by definition, TM; and its numerical value is

$$MT = OT - OM;$$

but, from equation (2), $OT = \frac{a^2}{x_1}$; and $OM = x_1$;

* The eccentric angle of any given point P on an ellipse is readily constructed thus: produce the ordinate MP to meet the major auxiliary circle in Q; the angle AOQ is the eccentric angle of the point P.

† The equations [55] are of great service in studying the ellipse by the methods of the differential calculus. They express the equations of the ellipse in terms of a single variable, ϕ.

hence $$MT = \frac{a^2}{x_1} - x_1,$$

i.e., $$TM = \frac{x_1^2 - a^2}{x_1}.$$

Hence the value of the subtangent, corresponding to any point of the ellipse whose equation is (1), depends only upon the major axis, and the abscissa of the point; therefore, if *a series of ellipses have the same major axis, tangents drawn to them at the points having a common abscissa will cut the major axis (extended) in a common point.*

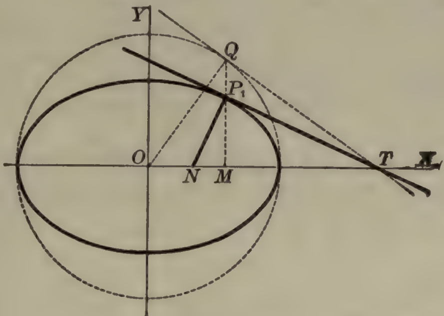

FIG. 84.

This fact suggests a method for constructing a tangent and normal to an ellipse, at a given point: draw the major auxiliary circle; at Q on this circle, and in MP_1 extended, draw a tangent to the circle. This will cut the axis in T; and P_1T will be the required tangent to the ellipse at P_1. The normal P_1N may then be drawn perpendicular to P_1T.

The equation of the normal through P_1 is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1);$$

therefore the x-intercept of the normal at that point is

$$ON = \frac{a^2 - b^2}{a^2} x_1 = e^2 x_1.$$

But the subnormal corresponding to P_1 is

$$NM = OM - ON$$

and $$OM = x_1$$

therefore $$NM = x_1 - \frac{a^2 - b^2}{a^2} x_1$$

$$= \frac{b^2}{a^2} x_1 = (1 - e^2) x_1.$$

Note. From the value of ON it follows that the normal to an ellipse does not, in general, pass through the center, but passes between the center and the foot of the ordinate; the extremities of the axes of the curve being exceptional points. If, however, $a = b$, then $e = 0$, the curve is a circle, and every normal passes through the center. [Cf. Art. 62.]

121. The tangent and normal bisect externally and internally, respectively, the angles between the focal radii of the point of contact.

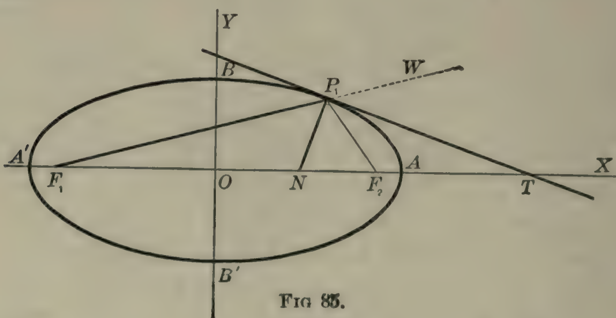

Fig 85.

Let the equation of the given ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; also let F_1 and F_2 be the foci, and $P_1 \equiv (x_1, y_1)$ any given point on the curve. Draw the tangent TP_1, the normal P_1N, and also the lines F_2P_1 and F_1P_1W

Then $$F_1N = F_1O + ON = ae + e^2x_1 \qquad \text{[Art. 120]}$$
$$= e(a + ex_1),$$
$$NF_2 = OF_2 - ON = ae - e^2x_1$$
$$= e(a - ex_1);$$
but $$F_1P_1 = a + ex_1 \qquad \text{[Art. 117]}$$
and $$F_2P_1 = a - ex_1.$$
Hence $$F_1N : NF_2 = F_1P_1 : P_1F_2;$$

and, by a theorem of plane geometry, this proportion proves that the normal P_1N bisects the angle $F_1P_1F_2$ between the focal radii. Again, since the tangent is perpendicular to the normal, the tangent P_1T will bisect the external angle F_2P_1W.

This proposition leads to a second method of constructing the tangent and normal to an ellipse at a given point. [Cf. Art. 120.] First determine the foci, F_1 and F_2 [Art. 117], then draw the focal radii to the given point and bisect the angle thus formed, — internally for the normal, externally for the tangent.

EXERCISES

1. Two tangents, only, may be drawn from a point to an ellipse. Prove; and discuss exceptional cases.

2. There are two tangents to an ellipse parallel to any given line.

3. Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at the focus.

4. Each of the two tangents drawn to the ellipse from a point on the directrix subtends a right angle at the focus.

5. The rectangle formed by the perpendiculars from the foci upon any tangent is constant; it is equal to the square of the semi-minor axis.

6. The circle on any focal distance as diameter touches the major auxiliary circle.

7. The perpendicular from the focus upon any tangent, and the line joining the center to the point of contact, meet upon the directrix.

8. The perpendicular from either focus, upon the tangent at any point of the major auxiliary circle, equals the distance of the corresponding point of the ellipse from that focus.

9. The latus rectum is a third proportional to the major and minor axes.

10. The area of the ellipse is πab.

> SUGGESTION. Employ the fact, proved in Art. 119, that the ordinate of an ellipse is to the corresponding ordinate of the major auxiliary circle as $b:a$, and thus compare the area of the ellipse with that of its major auxiliary circle.

122. Diameters. As already shown in Art. 108, the definition of a diameter as the locus of the middle points of a system of parallel chords leads directly to its equation, for the ellipse: *viz.*, Eq. [52]

$$y = -\frac{b^2}{a^2m}x.$$

The form of this equation shows that *every diameter of the ellipse passes through the center*

123. Conjugate diameters. Since every diameter passes through the center of the ellipse, and since, by varying the slope m of the given set of parallel chords, the corresponding diameter may be made to have any required slope, therefore it follows that *every chord that passes through the center of an ellipse is a diameter*, corresponding to some set of parallel

chords. In particular, that one of the given set of chords which passes through the center, *i.e.*, the chord whose equation is

$$y = mx, \tag{56}$$

is a diameter. This diameter bisects the chords parallel to the line [52]; for if m' be the slope of the line [52],

then $$m' = -\frac{b^2}{a^2m},$$

hence, $$mm' = -\frac{b^2}{a^2}; \tag{57}$$

and this equation expresses the condition that line [52], which has the slope m', shall bisect the chords of slope m [Art. 122]. But conversely, it expresses also the condition that the line [56] which has the slope m shall bisect the chords of slope m' Hence, each of the lines [52] and [56] bisects the chords parallel to the other. Hence, *if one diameter bisects the chords parallel to a second, then also the second diameter bisects the chords parallel to the first.* Such diameters are called **conjugate** to each other.

Each line of the set of parallel chords in general cuts the ellipse in two distinct points, and the farther the chord is from the center, the nearer these two points are to each other, and to their midpoint. In the limiting position, the chord becomes a tangent, with the two intersection points and their midpoint coincident at the point of tangency. Therefore, *the tangent at the end of a diameter is parallel to the conjugate diameter.* This property, with that of Art. 122, suggests a method for constructing conjugate diameters: first draw a tangent at an extremity of a given diameter [Art. 120], then a line drawn parallel to this tangent through the center of the ellipse is the required conjugate diameter. (See Fig. 76.)

124. Properties of conjugate diameters of the ellipse. (a) It has been seen [Art. 123] that two diameters are conjugate when their slopes satisfy the relation

$$mm' = -\frac{b^2}{a^2} \quad (1)$$

It follows, since the product of their slopes is negative, that with the exception of the case where one diameter is the minor axis itself, *conjugate diameters do not both lie in the same quadrant formed by the axes of the curve.*

(β) From the definition [Art. 123] it is evident that the minor and major axes of the ellipse are a pair of conjugate diameters, and they are at right angles to each other. Perpendicular lines, however, in general, fulfill the condition

$$mm' = -1; \quad (2)$$

hence, in general, equation (2) is not consistent with equation (1) for other values of m and m' than 0 and ∞, — the slopes for the axes of the curves. Hence, *the major and minor axes of the ellipse are the only pair of conjugate diameters that are perpendicular to each other.*

But for $b^2 = a^2$, *i.e.*, for the circle, it is clear that every pair of conjugate diameters satisfies equation (2), and that such diameters are therefore perpendicular to each other.

EXERCISES

1. Find the diameter of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which bisects the chords parallel to the line $3x - 5y + 10 = 0$.

2. Find the diameter conjugate to that of exercise 1.

3. Show that the lines $x - 4y = 0$, $2x + y = 0$ are conjugate diameters of the ellipse $x^2 + 2y^2 = 4$.

4. For the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, write the equations of diameters conjugate to the line

$$(\alpha)\ ax = by, \quad (\beta)\ bx = ay.$$

5. Show that the pair of diameters drawn parallel to the chords joining the extremities of the axes are equal and conjugate.

6. Two conjugate diameters of the ellipse $\frac{y^2}{25}+\frac{x^2}{4}=1$ have the slopes $\frac{5}{2}$ and $-\frac{5}{2}$, respectively; find their lengths.

7. Given the ellipse $x^2+5y^2=5$, find the eccentric angle for the point whose abscissa is -1. Also find the diameter conjugate to the one passing through this point.

8. Find the point of the ellipse $4x^2+9y^2=36$ corresponding to an eccentric angle of 45°, and also find the diameter conjugate to the one passing through this point.

9. Find the lengths of the diameters in Ex. 8.

10. Given $x^2+4y^2=12$; find the points on the major auxiliary circle corresponding to the ends of the latera recta, and also their eccentric angles.

REVIEW EXERCISES

1. Find the foci, directrices, eccentricity of the ellipse $3x^2+4y^2=12$.

2. Find the area of the ellipse $mx^2+ny^2=m^2n^2$.

3. For the ellipse $Ax^2+By^2-C=0$ show that the eccentricity is $e=\sqrt{1-\frac{A}{B}}$, (if $B>A$).

4. Find the locus of the foot of the perpendicular drawn from the center of the ellipse $x^2+4y^2=4$ to a variable tangent $y=mx+\sqrt{4m^2+1}$.

5. Prove that the line joining the point (3, 2) to the origin bisects the segment of the line $3x+8y=4$ intercepted by the ellipse $x^2+4y^2=4$.

6. Find the locus of the center of a circle which passes through the point (3, 0) and touches internally the circle $x^2 + y^2 = 100$.

7. Find the length of the minor axis of an ellipse whose major axis is 50 and whose area is equal to that of a circle whose radius is 10.

8. One focal radius is 6, the other is 3, for a point whose abscissa is 3; find the major axis of the ellipse, distance between foci, and area. answer in book wrong

9. A line of fixed length moves so that its ends remain in the coördinate axes; find the locus generated by any point of the line.

10. Find the locus of the middle points of chords of an ellipse drawn through the positive end of the minor axis.

11. With a given focus and directrix a series of ellipses are drawn; show that the locus of the extremities of their minor axes is a parabola.

12. Show that the line $x \cos \alpha + y \sin \alpha = p$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.$$

13. Find the locus of the foot of the perpendicular drawn from the center of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a variable tangent.

14. Prove, analytically, that if the normals to an ellipse pass through its center, the ellipse is a circle.

15. Find the locus of the vertex of a triangle of base $2a$, and such that the product of the tangents of the angles at its base is $\frac{b^2}{c^2}$.

16. The ratio of the subnormals for corresponding points on the ellipse and major auxiliary circle is $\frac{b^2}{a^2}$.

17. From an external point T whose coördinates are (h, k) a line is drawn to the center C meeting the ellipse in R; find the ratio of CT to CR.

18. Normals at corresponding points on the ellipse, and on the major auxiliary circle, meet on the circle $x^2 + y^2 = (a + b)^2$

The tangents drawn from point P to an ellipse make angles θ_1 and θ_2 with the major axis; find the locus of P:

19. When $\theta_1 + \theta_2 = 2\alpha$, a constant.

20. When $\tan\theta_1 + \tan\theta_2 = c$, a constant.

THE HYPERBOLA, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

125. Recapitulation. From the discussion already given of the hyperbola, and its close analogy to the ellipse, the following *data* may be assumed here:

The standard equation,	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$;
the eccentricity,	$e^2 = \frac{a^2 + b^2}{a^2}$; and $e > 1$,
the foci,	$F_1 \equiv (-ae, 0)$, $F_2 \equiv (ae, 0)$;
the directrices,	$x = -\frac{a}{e}$, $x = \frac{a}{e}$;
the tangent at P_1,	$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$;
the tangent with slope m,	$y = mx \pm \sqrt{m^2a^2 - b^2}$;
the diameter,	$y = \frac{b^2}{a^2 m}x$,

bisecting chords whose slope is m.

In spite of the close similarity between these results for the hyperbola and the ellipse, the nature of the hyperbola apparently differs widely from that of the ellipse, consisting, as it does, of two open infinite branches instead of one closed oval. It is desired now to show some of the most important properties of the hyperbola which correspond to similar properties in the ellipse; and also to prove some special properties which are peculiar to the hyperbola.

126. The difference between the focal distances of any point on an hyperbola is constant; it is equal to the transverse axis.

Let $P_1 \equiv (x_1, y_1)$ be any given point on the curve; it may be shown as in Art. 117 that the focal distances are

$$F_1P_1 = ex_1 + a, \quad (1) \qquad F_2P_1 = ex_1 - a, \quad (2)$$

where $$e > 1.$$

Subtracting equation (2) from equation (1) gives

$$F_1P_1 - F_2P_1 = 2\,a;$$

hence, *the difference between the focal distances of any point on an hyperbola is constant; it is equal to the transverse axis.*

If the foci are not given, they may be constructed as follows, provided the semi-axes of the curve are known: plot the points $A \equiv (a, 0)$ and $B \equiv (0, b)$; then with the center of the hyperbola as center, and the distance AB as radius, describe a circle; it will cut the transverse axis in the required foci F_1 and F_2, for $$AB = \sqrt{a^2 + b^2} = \sqrt{a^2e^2} = \pm\, ae.$$

127. Construction of the hyperbola. The property of the preceding article might be taken as a new definition of the hyperbola, viz., *the hyperbola is the locus of a point the difference of whose distances from two fixed points is constant.* This definition leads at once to the equation of the curve [cf. Ex. 33, p. 57], and also to a method for its construction.

(α) *Construction by separate points.* Let $A'A$ be the given difference of the focal distances, — *i.e.*, the transverse axis of the hyperbola, — and F_1 and F_2 the given fixed points, the foci. With either focus, say F_1, as a center, and a radius $A'R \gtreqless A'A$, describe an arc; then with the other focus as a center, and a radius AR describe an arc cutting the first arcs in the two points P_1. These are points of the hyperbola. Similarly, as many points as desired may be obtained and then connected by a smooth curve, — approximately an hyperbola.

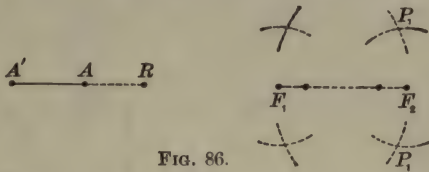

FIG. 86.

(β) *Construction by a continuously moving point; the foci being given.* Pivot a straightedge LM at one focus F_1, so that F_1M is greater than the transverse axis $2a$; at M and the other focus F_2 fasten the ends of a string of length l, such that $F_1M = l + 2a$; then a pencil P held against the string and straightedge (see Fig. 87), so as to keep the string always taut, will, while the straightedge revolves about F_1, trace one branch of the hyperbola. By fastening the string at the first focus and the straightedge at the second, the other branch of the curve can be traced.

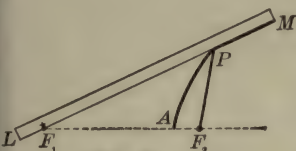

FIG. 87.

128. The tangent and normal bisect internally and externally the angles between the focal radii of the point of contact.

Let F_1 and F_2 be the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, P_1T the tangent, and P_1N the normal at the point $P_1 \equiv (x_1, y_1)$.

Then the equation of P_1T is $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$, and the length of the intercept OT of the tangent is

$$OT = \frac{a^2}{x_1}.$$

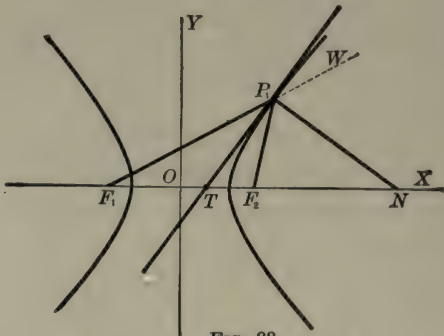

FIG. 88.

Now, in the triangle $F_1P_1F_2$,

$$F_1T = F_1O + OT = ae + \frac{a^2}{x_1}$$

$$= \frac{a}{x_1}(ex_1 + a),$$

and
$$TF_2 = OF_2 - OT = ae - \frac{a^2}{x_1}$$

$$= \frac{a}{x_1}(ex_1 - a);$$

but
$$F_1P_1 = ex_1 + a, \qquad \text{[Art. 126]}$$

and
$$P_1F_2 = ex_1 - a.$$

Hence $F_1T : TF_2 = F_1P_1 : P_1F_2,$

and, by elementary geometry, the tangent bisects internally the angle between the focal radii. Then, since the normal is perpendicular to the tangent, the normal P_1N bisects the external angle F_2P_1W.

These facts suggest a method, analogous to that of Art. 121, for constructing the tangent and normal to an hyperbola at a given point.

129. Conjugate hyperbolas. A curve bearing very close relations to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{1}$$

is that represented by the equation

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

i.e., by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \tag{2}$$

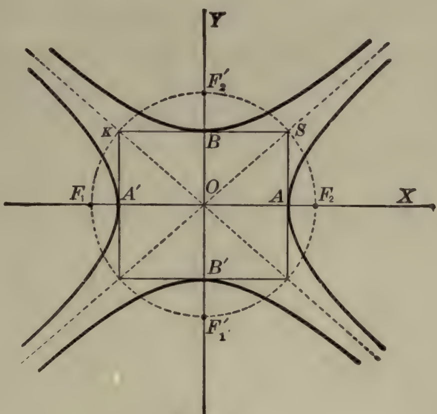

FIG. 89.

in which a and b have the same values as in equation (1). This curve is evidently an hyperbola, and has for its transverse and conjugate axes, respectively, the conjugate and transverse axes of the original, or primary, hyperbola. Two

such hyperbolas are called **conjugate** hyperbolas; they are sometimes spoken of as the x- and y-hyperbolas, respectively.

It follows at once that the hyperbola (2), conjugate to the hyperbola (1), has for its eccentricity

$$e' = \frac{\sqrt{a^2 + b^2}}{b},$$

for foci the points $(0, \pm be')$, and for directrices the lines

$$y = \pm \frac{b}{e'}.$$

Two conjugate hyperbolas have a common center, and their foci are all at the common distance $\sqrt{a^2 + b^2}$ from this center; *i.e.*, the foci all lie on a circle about the center, having for radius the semi-diagonal OS of the rectangle upon their common axes, the sides of which are tangent to the curves at their vertices. Moreover, when the curves are constructed it will be found that they do not intersect, but are separated by the diagonals OS and OK, extended, of this circumscribed rectangle, which they approach from opposite sides. These diagonals are examples of a class of lines of great interest in analytic theory; they are called *asymptotes*.

EXERCISES

1. Construct an hyperbola, with vertex at the point (3, 0) and focus at the point (5, 0).

2. Construct an hyperbola, given the distance from directrix to focus as 4 cm. How many such hyperbolas are possible?

3. Write the equation of an hyperbola conjugate to the hyperbola $9y^2 - 16x^2 = 144$, and find its axes, foci, and latus rectum. Sketch the figure.

4. Write the equations of the tangent and normal to the hyperbola $3x^2 - 4y^2 = 11$ at the point $(-3, -2)$, and find the subtangent and subnormal.

5. For what points of an hyperbola are the subtangent and subnormal equal?

6. Given the hyperbola $9y^2 - 4x^2 = 36$, find the focal radii of the ends of the latera recta.

7. A tangent which is parallel to the line $5x + 4y + 10 = 0$, is drawn to the hyperbola $x^2 - y^2 = 9$; what is the subnormal for the point of contact?

8. What tangent to the hyperbola $\frac{x^2}{10} - \frac{y^2}{12} = 1$ has its x-intercept 4?

9. Find the two tangents to the hyperbola $4x^2 - 2y^2 = 6$ which are drawn through the point $(0, 5)$.

10. Prove that if the crack of a rifle and the thud of the ball on the target are heard at the same instant, the locus of the hearer is an hyperbola.

11. Find the equation of an hyperbola whose vertex bisects the distance from the focus to the center.

12. If e and e' are the eccentricities of an hyperbola and its conjugate, then $$e^2 + e'^2 = e^2e'^2.$$

13. If e and e' are the eccentricities of two conjugate hyperbolas, then $$ae = be'.$$

14. Find the eccentricity and latus rectum of the hyperbola
$$x^2 = 4(y^2 + a^2).$$

15. Find the tangents to the hyperbola $4x^2 - 15y^2 = 60$, which, with the tangent at the vertex, form a circumscribed equilateral triangle. Find the area of the triangle.

16. Find the lengths of the tangent, normal, subtangent, and subnormal for the point $(2, 3)$ of the hyperbola $3x^2 - y^2 = 3$.

130. Asymptotes. If a tangent to an infinite branch of a curve approaches more and more closely to a fixed straight line as a limiting position, when the point of contact moves farther and farther away on the curve, and becomes infinitely distant, then the fixed line is called an **asymptote** of the curve.* More briefly, though less accurately, this definition may be stated as follows: an asymptote to a curve is a tangent whose point of contact is at infinity, but which is not itself entirely at infinity. It is evident that to have an asymptote a curve must have an infinite branch; and this branch may be considered as having two coincident, and infinitely distant, points of intersection with its asymptote. This property will aid in obtaining the equation of the asymptote.

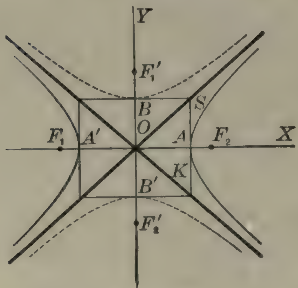

FIG. 90.

The hyperbola $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{1}$$

is cut by the line $$y = mx + c, \tag{2}$$

in points whose abscissas are given by the equation

$$(a^2m^2 - b^2)x^2 + 2\,a^2cmx + a^2b^2 + a^2c^2 = 0. \tag{3}$$

If line (2) is an asymptote, the roots of equation (3) must both be infinite; therefore, by Art. 7, (4),

$$a^2m^2 - b^2 = 0 \quad \text{and} \quad 2\,a^2cm = 0, \tag{4}$$

hence $$m = \pm\frac{b}{a} \quad \text{and} \quad c = 0.$$

---

* This definition implies that the distance between a curve and its asymptote becomes infinitely small. McMahon & Snyder, "Differential Calculus," Chap. XIV.

Substituting these values in equation (2) gives

$$y=\frac{b}{a}x, \quad \text{and} \quad y=-\frac{b}{a}x; \tag{5}$$

and these equations represent the asymptotes of the hyperbola: they are the lines OS and OK in Fig. 90.

Therefore, *the hyperbola has two asymptotes; they pass through its center, and are the diagonals of the rectangle described upon its axes.*

Since the equation of the hyperbola conjugate to (1) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \tag{6}$$

and thus differs from equation (1) only in the sign of the second member, which affects only the constant term in equation (3), therefore the equations (4) determine the value of m and c for the asymptotes of the conjugate hyperbola also. It follows that *conjugate hyperbolas have the same asymptotes.*

The equations of the asymptotes may be combined, by Art. 37, into the one equation which represents both lines, *viz.*,

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.} \tag{58}$$

131. Relation between conjugate hyperbolas and their asymptotes. It has been seen that the standard forms for the equations of the primary hyperbola, its asymptotes, and its conjugate hyperbola are, respectively,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{1}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \tag{2}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \tag{3}$$

It will be noticed at once that these three equations differ only in their constant terms; and that the equation of the primary hyperbola (1) differs from that of the asymptotes (2) by the negative of the constant by which the equation of the conjugate hyperbola (3) differs from equation (2). Moreover, this relation between the equations of the three loci must hold when not in their standard forms, *i.e.*, whatever the coördinate axes. For, any transformation of coördinates will affect only the first member of equations (1), (2), and (3), and will affect these in precisely the same way. After the transformation, therefore, the equations of the loci will differ only by a constant (not usually by 1, however, when cleared of fractions); and the value of the constant in the equation of the asymptotes will be midway between the values of the constants in the equations of the two hyperbolas.

EXAMPLE 1. An hyperbola having the lines

(1) $x+2y+3=0$ and (2) $3x+4y+5=0$

for asymptotes, will have an equation of the form

$$(x+2y+3)(3x+4y+5)+k=0, \tag{3}$$

while the equation of its conjugate hyperbola will be

$$(x+2y+3)(3x+4y+5)-k=0. \tag{4}$$

If a second condition is imposed upon the hyperbola, *e.g.*, that it shall pass through the point $(1, {}^-1)$, then the value of k may be easily found thus: since the curve passes through the point $(1, {}^-1)$, therefore by equation (3),

$$(1-2+3)(3-4+5)+k=0; \therefore k=-8,$$

and the equation of the hyperbola is

$$(x+2y+3)(3x+4y+5)-8=0,$$

that is, $3x^2+10xy+8y^2+14x+22y+7=0$; (5)

and the equation of the conjugate hyperbola

$$3x^2 + 10xy + 8y^2 + 14x + 22y + 23 = 0.$$

EXAMPLE 2. If the asymptotes of an hyperbola are chosen as the coördinate axes, their equations will be $x = 0$ and $y = 0$, respectively; or, combined in one equation,

$$xy = 0. \tag{1}$$

Therefore the equation of the hyperbola, — which differs from that of its asymptotes by a constant, — is

$$xy = k, \tag{2}$$

wherein the value of the constant k is to be determined by an additional assigned condition concerning the curve; *e.g.*, that it shall pass through a point, such as the vertex,

$$\left(\frac{\sqrt{a^2+b^2}}{2}, \frac{\sqrt{a^2+b^2}}{2}\right).$$

The equation becomes

$$xy = \frac{a^2+b^2}{4}, \tag{3}$$

which is the equation of the hyperbola referred to its asymptotes as axes, ordinarily oblique.

132. Equilateral or rectangular hyperbola. If the axes of an hyperbola are equal, so that $a = b$, its equation has the form

$$x^2 - y^2 = a^2, \tag{1}$$

and its eccentricity is $e = \sqrt{2}$. Its conjugate hyperbola has the equation

$$x^2 - y^2 = -a^2; \tag{2}$$

with the same eccentricity and the same shape; while its asymptotes have the equations

$$x = \pm y, \tag{3}$$

and are therefore the bisectors of the angles formed by the axes of the curves; hence the asymptotes of these hyperbolas are perpendicular to each other. The hyperbola whose axes are equal is therefore called an **equilateral**, or a **rectangular hyperbola**, according as it is thought of as having equal axes or asymptotes at right angles.

EXERCISES

1. Find the asymptotes of the hyperbola $9x^2-16y^2=144$, and the angle between them.

2. If the vertex lies one third of the distance from the center to the focus, find the equations of the hyperbola, and of its asymptotes.

3. If a line $y=mx+c$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in one finite and one infinitely distant point, the line is parallel to an asymptote.

4. Show that, in an equilateral hyperbola, the distance of a point from the center is a mean proportional between its focal distances.

5. Find the equation of the hyperbola passing through the point (1, 0), and having for asymptotes the lines

$$x+y+3=0 \quad \text{and} \quad x-y+7=0$$

6. Write the equation of the hyperbola conjugate to that of Ex. 5.

7. Find the equations of the asymptotes of the hyperbola

$$x^2-4y^2-2x-16y-19=0;$$

also find the equation of the conjugate hyperbola.

8. Find the equation of the asymptotes of the hyperbola

$$7x^2+50xy+7y^2=50$$

9. Find the equation of the hyperbola conjugate to

$$9y^2 - 4x^2 - 8x + 18y + 41 = 0.$$

10. Prove that a perpendicular from the focus to an asymptote of an hyperbola is equal to the semi-conjugate axis.

11. The asymptotes meet the directrices of the x-hyperbola on the x-auxiliary circle, and of the conjugate hyperbola on the y-auxiliary circle.

12. The circle described about a focus, with a radius equal to half the conjugate axis, will pass through the intersections of the asymptotes and a directrix.

13. The line from the center C to the focus F of an hyperbola is the diameter of a circle that cuts an asymptote at P; show that the chords CP and FP are equal, respectively, to the semi-transverse and semi-conjugate axes.

133. Diameters. The equation of the diameter to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is

$$y = \frac{b^2}{a^2 m} x. \qquad [53]$$

This equation shows that *every diameter of the hyperbola passes through the center.*

Conversely, it is true, as in the case of the ellipse, that every chord of the hyperbola through the center is a diameter. That chord of the original set which passes through the center is the diameter *conjugate* to [53]; and its equation is

$$y = mx. \qquad [59]$$

Let m' be the slope of a diameter, and m that of its conju-

gate; the essential condition that two diameters should be conjugate to each other is that [cf. Art. 123]

$$mm' = \frac{b^2}{a^2} \qquad [60]$$

134. Properties of conjugate diameters of the hyperbola.

(α) It is clear that the condition [60] holds also for the hyperbola

$$\frac{x^2}{-a^2} + \frac{y^2}{b^2} = 1,$$

which is conjugate to the given hyperbola; for, replacing a^2 by $-a^2$ and $-b^2$ by b^2 leaves equation [60] unchanged. Hence, *diameters which are conjugate to each other for a given hyperbola are conjugates also for the conjugate of that hyperbola.*

(β) The axes of the hyperbola are clearly diameters of the curve. They are perpendicular to each other, and therefore satisfy the relation

$$mm' = -1.$$

Compare this condition with that of equation [60]; it follows that *the transverse and conjugate axes of the hyperbola are the only pair of perpendicular conjugate diameters* [cf. (β), Art. 124].

If $a = b$, the condition [60] reduces to

$$mm' = 1;$$

therefore [Art. 12], in the rectangular hyperbola the sum of the angles which a pair of conjugate diameters make with the transverse axis is 90°.

(γ) Since in equation [60] the product mm' is positive, it follows that the angles which conjugate diameters make with the transverse axis are both acute, or both obtuse. Moreover,

if $$m < \pm \frac{b}{a}, \quad \text{then} \quad m' > \pm \frac{b}{a};$$

and the diameters lie on opposite sides of an asymptote; *i.e.*, *two conjugate diameters lie in the same quadrant formed by the axes of the hyperbola and on opposite sides of the asymptote.* [Cf. Art. 124, (α)].

(δ) An asymptote passes through the center of an hyperbola, hence may be regarded as a diameter. Its slope is

$$m = \pm \frac{b}{a}, \quad \therefore \ m' = \pm \frac{b}{a};$$

hence, *an asymptote regarded as a diameter is its own conjugate;* it may be called a *self-conjugate* diameter.

This is a limiting case of (γ) above.

(ϵ) It follows from this last fact that if a diameter intersects a given hyperbola, then the conjugate diameter does not intersect it, but cuts the conjugate hyperbola. It is customary and useful to define as the extremities of the conjugate diameter its points of intersection with the conjugate hyperbola. With this limitation, it follows from (α) of this article that, as in the ellipse, *each of two conjugate diameters bisects the chords parallel to the other.*

(ζ) As a limiting case of this last proposition, also, it is evident that *the tangent at the end of a diameter is parallel to the conjugate diameter.*

By reasoning entirely analogous to that given in Art. 124, for the ellipse, properties similar to those there given may be derived for the hyperbola. They are included in the following exercises, to be worked out by the student.

EXERCISES

1. Find the equation of the diameter of the hyperbola $4x^2 - 5y^2 = 20$ which bisects the chords $y = -4x + b$. Find also the conjugate diameter.

2. Find, for the hyperbola of Ex. 1, a diameter through the point $(^-1, ^-1)$; also find its conjugate diameter.

3. Find the diameter of the hyperbola $9x^2 - 16y^2 = 25$ which is conjugate to the diameter $x = 5y$.

4. Find the equation of a chord of the hyperbola $4x^2 - 3y^2 = 36$, which is bisected at the point $(^-8, ^-4)$.

5. Show that, in an equilateral hyperbola, conjugate diameters make equal angles with the asymptotes.

6. The difference of the squares of two conjugate semi-diameters is constant; it is equal to the difference of the squares of the semi-axes.

7. The angle between two conjugate diameters is $\sin^{-1}\dfrac{ab}{a'b'}$.

8. Tangents at the ends of a pair of conjugate diameters intersect on an asymptote.

REVIEW EXAMPLES

1. Write the equation of an hyperbola whose transverse axis is $2\sqrt{3}$ and the eccentricity equal to 2.

2. Find the equation of the diameter of the hyperbola $16x^2 - 9y^2 = 144$, the coördinates of the extremity of a diameter being $(4, \frac{20}{3})$. Find the equation of the conjugate diameter.

3. Assume the equation of the hyperbola, and show that the difference of the focal distances is constant.

4. Find the locus of the vertex of a triangle of given base $2c$, if the difference of the two other sides is a constant, and equal to $2a$.

5. Find the locus of the vertex of a triangle of given base, if the difference of the tangents of the base angles is constant.

6. Find the asymptotes of the hyperbola $xy - 4x - 5y = 0$. What is the equation of the conjugate hyperbola?

7. Show that two concentric rectangular hyperbolas, whose axes meet at an angle of 45°, cut each other orthogonally.

8. The portions of any chord of an hyperbola intercepted between the curve and its conjugate are equal.

SUGGESTION. Draw a tangent parallel to the line in question.

9. For the hyperbola $Ax^2 - By^2 + C = 0$ show that $e = \sqrt{1 + \frac{A}{B}}$. Is there any restriction upon the values of A and B? [Cf. Ex. 3, p. 207].

10. Prove that the asymptotes of the hyperbola $xy + 4y + 4x = 0$ are $x + 4 = 0$ and $y + 4 = 0$.

11. Find the coördinates of the points of contact of the common tangents to the hyperbolas,

$$4x^2 - 4y^2 = 3a^2, \text{ and } 2xy = a^2.$$

12. If a right-angled triangle be inscribed in a rectangular hyperbola, prove that the tangent at the right angle is perpendicular to the hypotenuse.

13. Show that the line $y = mx + 2k\sqrt{-m}$ always touches the hyperbola $xy = k^2$; and that its point of contact is $\left(\frac{k}{\sqrt{-m}}, k\sqrt{-m}\right)$.

14. If an ellipse and hyperbola have the same foci, they intersect at right angles.

15. Find tangents to the hyperbola $x^2 - 4y^2 = 1$ which are perpendicular to its asymptotes.

16. Find normals to the hyperbola $\frac{(x-3)^2}{16} - \frac{(y-2)^2}{9} = 1$ which are parallel to its asymptotes.

17. Show that in an equilateral hyperbola conjugate diameters are equally inclined to the asymptotes.

18. Show that two conjugate diameters of a rectangular hyperbola are equal.

19. Find the equation of the tangent to $xy = 4$ at the point (x_1, y_1) by the secant method.

20. Show that, in an equilateral hyperbola, two diameters at right angles to each other are equal.

21. A variable circle is always tangent to each of two fixed circles; prove that the locus of its center is either an hyperbola or an ellipse.

22. Find the common tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its mid-circle $x^2 + y^2 = ab$.

23. In the hyperbola $25x^2 - 16y^2 = 400$, find the conjugate diameters that cut each other at an angle of 45°.

24. The latus rectum of an hyperbola is a third proportional to the two axes.

25. Find the equation of a line through the focus of an hyperbola and the focus of its conjugate.

26. Show that the x-axis is an asymptote of the hyperbola

$$2xy + 3y^2 + 4y = 9.$$

What is the equation of the other asymptote? Of the conjugate hyperbola?

27. If two tangents are drawn from an external point to an hyperbola, they will touch the same or opposite branches of the curve according as the given point lies between or outside of the asymptotes.

28. If P be the middle point of a line AB which is so drawn as to cut off a constant area from the corner of a square, the locus of P is an equilateral hyperbola.

PART II

SOLID ANALYTIC GEOMETRY

CHAPTER X

COÖRDINATE SYSTEMS. THE POINT

135. Solid Analytic Geometry treats by analytic methods problems which concern figures in space, and therefore involves three dimensions. It is evident that new systems of coördinates must be chosen, involving three variables; and that the analytic work will therefore be somewhat longer than in the plane geometry. On the other hand, since a plane may be considered as a special case of a solid where one dimension has the particular value zero, it is to be expected that the analytic work with three coördinate variables should be entirely consistent with that for two variables, — merely a simple extension of the latter. The student should not fail to notice this close analogy in all cases.

136. Rectangular coördinates. Let three planes be given fixed in space and perpendicular to each other, — the **coördinate planes** XOY, YOZ, and ZOX. They will intersect by pairs in three lines, $X'X$, $Y'Y$, and $Z'Z$, also perpendicular to each other, called the **coördinate axes.** These three lines will meet in a common point O, called the **origin.** Any three other

planes, LP, MP, and NP, parallel respectively to these coördinate planes, will intersect in three lines, $N'P$, $L'P$, $M'P$, which will be parallel respectively to the axes; and these three lines will meet in, and completely fix the position of, a point P in space. The directed distances $N'P$, $L'P$, and $M'P$ thus determined, *i.e.*, the perpendicular distances of the point P from the coördinate planes, are the **rectangular coördinates** of the point P. They are represented respectively by x, y, and z.

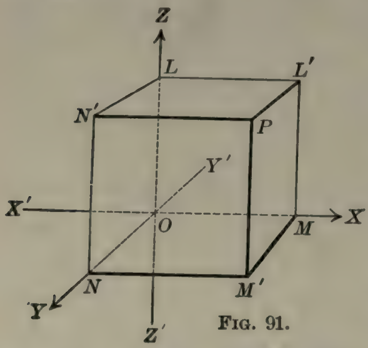

FIG. 91.

It is clear that

$$x = N'P = LL' = NM' = OM;$$
$$y = L'P = MM' = LN' = ON;$$
$$z = M'P = NN' = ML' = OL.$$

It is generally convenient, however, to consider

$$x = OM, \quad y = MM', \quad \text{and} \quad z = M'P.$$

The point may be denoted by the symbol $P \equiv (x, y, z)$.

The axes may be directed at pleasure; it is usual to take the positive directions as shown in the figure. Then the eight portions, or octants, into which space is divided by the coördinate planes, will be distinguished completely by the signs of the coördinates of points within them.

If the chosen coördinate planes were oblique to each other, a set of oblique coördinates for any point in space might be found in an entirely analogous way.

Unless otherwise stated, rectangular coördinates will be used in the subsequent work.

137. Direction angles: direction cosines. In another useful method of fixing a point in space the axes of reference are chosen as in rectangular coördinates, and any point P of space is fixed by its distance from the origin, called the **radius vector**, and the angles α, β, γ, which this radius vector makes with the coördinate axes, respectively. These angles are called the **direction angles** of the line OP, and their cosines, its **direction cosines**. The point may be concisely denoted as the point $P \equiv (\rho, \alpha, \beta, \gamma)$.

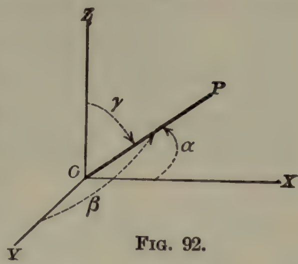

FIG. 92.

Simple equations connect these coördinates with those of the rectangular system; for, projecting OP upon the axes OX, OY, and OZ, respectively (cf. Figs. 91, 92),

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma. \qquad [61]$$

Again, $\overline{OP}^2 = \overline{OM'}^2 + \overline{M'P}^2 = \overline{OM}^2 + \overline{MM'}^2 + \overline{M'P}^2$,

i.e., $$\rho^2 = x^2 + y^2 + z^2. \qquad [62]$$

Moreover, the direction cosines are not independent, but are connected by an equation; for, by combining the above equations,

$$\rho^2 = \rho^2 \cos^2 \alpha + \rho^2 \cos^2 \beta + \rho^2 \cos^2 \gamma,$$

i.e., $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \qquad [63]$$

Such a relation was to have been expected, since only three magnitudes are necessary to determine the position of a point, and therefore the four numbers ρ, α, β, γ could not be independent.

Any three numbers, a, b, c, are proportional to the direction cosines of some line; because if these numbers are considered

as the coördinates of a point, then by equation [61] the direction cosines of the radius vector of that point are

$$\cos\alpha=\frac{a}{\sqrt{a^2+b^2+c^2}},\ \cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}},\ \cos\gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}. \quad [64]$$

These direction cosines are proportional to a, b, c; and are found by dividing a, b, and c, respectively, by the constant

$$\sqrt{a^2+b^2+c^2}.$$

Direction cosines are useful in giving the direction of any line in space. The direction of any line is the same as that of a parallel line through the origin, therefore the direction of a line may be given by the direction angles of some point whose radius vector is parallel to the line.

138. Distance and direction from one point to another. Let OX, OY, OZ be a set of rectangular axes, and $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be two given points. Then the planes through P_1 and P_2 parallel, respectively, to the coördinate planes, form a rectangular parallelepiped, of which the required distance P_1P_2 is a diagonal. From the figure,

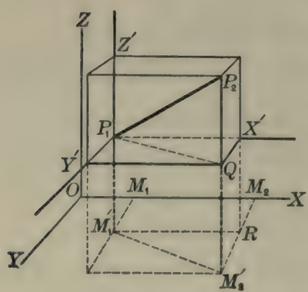

FIG. 93.

since $\angle P_1QP_2 = 90°$

and $\angle M_1'RM_2' = 90°$,

therefore $$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2 = \overline{M_1'M_2'}^2 + \overline{QP_2}^2$$
$$= \overline{M_1'R}^2 + \overline{RM_2'}^2 + \overline{QP_2}^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

That is, if d be the required distance,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad [65]$$

Moreover, since the direction of the line P_1P_2 is given by the angles α, β, γ, which it makes, respectively, with the lines P_1X', P_1Y', and P_1Z', drawn through P_1 parallel to the axes, therefore the projection of $d (= P_1P_2)$ upon these lines in turn gives

$$P_1P_2 \cos\alpha = P_1X', \quad P_1P_2 \cos\beta = P_1Y', \quad P_1P_2 \cos\gamma = P_1Z',$$

i.e., $d\cos\alpha = x_2 - x_1, \quad d\cos\beta = y_2 - y_1, \quad d\cos\gamma = z_2 - z_1;$

and, finally,

$$\cos\alpha = \frac{x_2 - x_1}{d}, \; \cos\beta = \frac{y_2 - y_1}{d}, \; \cos\gamma = \frac{z_2 - z_1}{d}. \qquad [66]$$

If d is taken positive, these equations determine the direction from P_1 to P_2.

139. Angle between two radii vectores. Angle between two lines. Let $P_1 \equiv (\rho_1, \alpha_1, \beta_1, \gamma_1)$ and $P_2 \equiv (\rho_2, \alpha_2, \beta_2, \gamma_2)$ be two given points, and θ the angle included by the radii vectores ρ_1 and ρ_2. Then, projecting upon OP_1 [Art. 13],

$$\text{proj. } OP_2 = \text{proj. } OM_2M_2'P_2,$$

i.e., $\rho_2 \cos\theta = OM_2 \cos\alpha_1 + M_2M_2' \cos\beta_1 + M_2'P_2 \cos\gamma_1.$

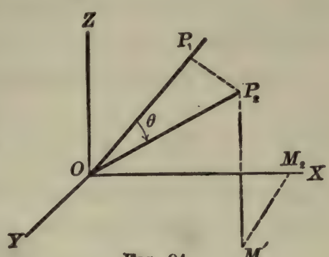

FIG. 94.

But $\qquad OM_2 = \rho_2 \cos\alpha_2,$

$\qquad M_2M_2' = \rho_2 \cos\beta_2,$

and $\qquad M_2'P_2 = \rho_2 \cos\gamma_2;$

hence,

$$\rho_2 \cos\theta = \rho_2 \cos\alpha_2 \cos\alpha_1 + \rho_2 \cos\beta_2 \cos\beta_1 + \rho_2 \cos\gamma_2 \cos\gamma_1;$$

i.e., $$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2, \qquad [67]$$

and this relation determines the required angle θ.

It follows, since any two straight lines in space have their directions given by the direction angles of radii vectores which are parallel to them, respectively, that formula [67] applies as well to the angle θ between any two straight lines in space, whose direction angles are given.

Two special cases arise, that of parallel and that of perpendicular lines. If the two given lines are parallel, evidently

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \gamma_1 = \gamma_2; \tag{68}$$

and formula [67] reduces to equation [63]. If the lines are perpendicular, $\cos\theta = 0$, and equation [67] reduces to

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0. \tag{69}$$

EXAMPLES ON CHAPTER X

1. Prove that the triangle formed by joining the points $(a, b, c,)$, (b, c, a), and (c, a, b), in pairs, is equilateral.

2. The direction cosines of a straight line are proportional to $^-3$, $^-2$, $^+1$; find their values. [Cf. Art. 137.]

3. Find the angle between two straight lines whose direction cosines are proportional to 3, 3, and 3, and 7, $^-4$, and 5, respectively.

4. Prove that if P_3 bisects the line P_1P_2

then
$$x_3 = \frac{x_1 + x_2}{2}, \quad y_3 = \frac{y_1 + y_2}{2}, \quad z_3 = \frac{z_1 + z_2}{2}. \tag{70}$$

5. Find the direction angles of a straight line which makes equal angles with the three coördinate axes.

6. A straight line makes the angle 30° with the x-axis, and 60° with the z-axis. What angle does it make with the y-axis?

7. Prove analytically that the straight lines joining the mid-points of the opposite edges of a tetrahedron pass through a common point, and are bisected by it.

8. What is the length of a line whose projections on the coördinate axes are 5, 6, 7, respectively?

9. Find the radius vector, and its direction cosines, for each of the points $(^-1, ^-2, ^-3)$, $(0, ^-1, ^-1)$, and (a, b, c).

10. Prove that if P_3 divides the line P_1P_2 in the ratio $\frac{m_1}{m_2}$, so that $P_1P_3 : P_3P_2 = m_1 : m_2$, [cf. Art. 24],

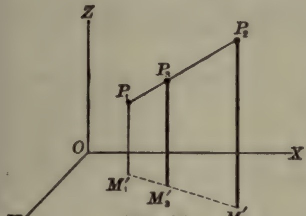

FIG. 95

then
$$\left.\begin{aligned} x_3 &= \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \\ y_3 &= \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \\ z_3 &= \frac{m_1z_2 + m_2z_1}{m_1 + m_2}. \end{aligned}\right\} \quad [71]$$

11. Find the coördinates of the points dividing the line from (0, 0, 0) to (3, 4, 5) externally and internally in the ratio 1 : 5.

12. Prove analytically that the straight lines joining the midpoints of the opposite sides of any quadrilateral bisect each other.

13. Show that the equations of transformation from one set of rectangular axes to a parallel set are

$$x = x' + h, \quad y = y' + k, \quad z = z' + j.$$

14. Show that the equations of transformation from one set of rectangular axes to another through the same origin, making the direction angles $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, and $\alpha_3, \beta_3, \gamma_3$ respectively with the old axes, are

$$x = x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3,$$
$$y = x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \gamma_3,$$
$$z = x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3.$$

CHAPTER XI

THE LOCUS OF AN EQUATION. SURFACES

140. Attention has been called to the close analogy between the corresponding analytical results for the geometry of the plane and of space. In geometry of one dimension, restricted to a line, the point is the elementary conception. Position is given by one variable, referring to a fixed point in that line; and any algebraic equation in that variable represents one or more points. In geometry of two dimensions, however, it has been shown that the line may be taken as the fundamental element. Position is given by two variables, referring to two fixed lines in the plane; and any algebraic equation in the two variables represents a curve, *i.e.*, a line whose generating point moves so as to satisfy some condition or law. Correspondingly, in geometry of three dimensions the surface is the elementary conception. Position is given by three variables, referring to three fixed surfaces, since any point is the intersection of three surfaces; and it can be shown that any algebraic equation in three variables represents some surface.

The study of the special equations of first and second degree in three variables will be taken up in the two succeeding chapters. Here it is desired to consider briefly three simple classes of surfaces: (1) **planes** parallel to the coördinate planes; (2) **cylinders,** *i.e.*, surfaces which are generated by

a straight line moving parallel to a fixed straight line, and always intersecting a fixed curve; and (3) **surfaces of revolution**, *i.e.*, surfaces generated by revolving some plane curve about a fixed straight line lying in its plane.

141. Equations in one variable. Planes parallel to coördinate planes. From the definition of rectangular coördinates it follows that the equations

$$x=0, \quad y=0, \quad z=0,$$

represent the coördinate planes, respectively, and that any algebraic equation in one variable and of the first degree, as $y=-3$, represents a plane parallel to one of them. Similarly an equation in one variable, and of degree n, will represent n such parallel planes, either real or imaginary. For, the first member of any such equation, as

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0, \tag{1}$$

can be factored into n linear factors, real or imaginary,

$$p_0(x-x_1)(x-x_2)(\cdots)(x-x_n)=0; \tag{2}$$

and by the reasoning of Art. 37, equation (2) will represent the loci of the n equations

$$x-x_1=0, \quad x-x_2=0, \quad \cdots, \quad x-x_n=0,$$

each of which is a plane, parallel to the yz-plane, and real if the corresponding root is real. In the same way, an equation in y or z only will represent planes parallel to the zx- or xy-plane.

Any algebraic equation in one variable represents one or more planes parallel to a coördinate plane.

It follows at once, by Art. 36, that two simultaneous equations of the first degree, each in one variable, represent the intersection of two planes parallel to coördinate planes; therefore, represent a straight line parallel to the coördinate axis of

the third variable; *e.g.*, $y = b$, $z = c$, considered as simultaneous equations, represent a straight line parallel to the x-axis.

142. Equations in two variables. Cylinders perpendicular to coördinate planes. Consider the equation

$$2x + 3y = 6, \tag{1}$$

in two variables only. In the xy-plane it represents a straight line AB. If, now, from any point P of AB a straight line be drawn parallel to the z-axis, the x and y coördinates of every point Q on this line will be the same as for P, and therefore satisfy equation (1). Moreover, if the line PQ be moved along AB, and always parallel to the z-axis, still the coördinates of every point in it satisfy equation (1). As the line PQ is thus moved, it traces a plane surface perpendicular to the xy-plane; and, as the coördinates of a point not on this surface evidently do not satisfy equation (1), this plane is the locus of equation (1).

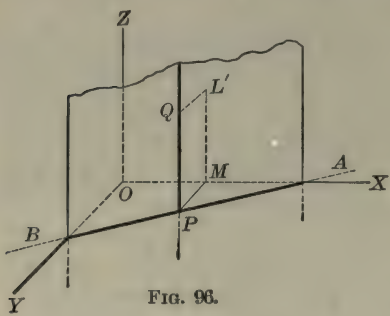

FIG. 96.

Again: the equation $\qquad y^2 + z^2 = r^2 \qquad (2)$

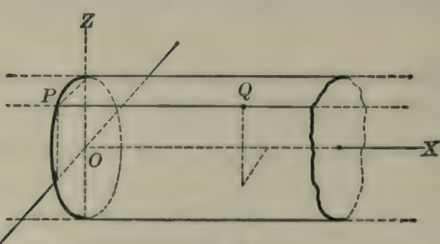

FIG. 97.

represents in the yz-plane a circle. It is therefore satisfied by the coördinates of any point Q, in a line parallel to the x-axis, through any point P of this circle; and also by the coördinates of Q as this line PQ is moved along the circle and parallel to the x-axis. The circular cylinder thus traced by the line PQ, perpendicular to the yz-plane, is the locus of the given equation.

Similarly, it may be shown that the locus of the equation

$$\frac{x^2}{a^2}-\frac{z^2}{b^2}=1, \tag{3}$$

is a cylindrical surface traced by a straight line parallel to the y-axis, and moving along the hyperbola whose equation in the xz-plane is equation (3). And in general, it is clear by analogy that *any algebraic equation in two variables represents a surface whose elements are parallel to the axis of the third variable, and which has its form and position determined by the plane curve represented by the same equation.* Such a surface is **cylindrical.**

If, now, a cylinder has its axis parallel to a coördinate axis, a section made by a plane perpendicular to that axis is a curve parallel to, and equal to, the directing curve on the coördinate plane, and is represented in the cutting plane by the same equation.

> Thus, the section of the elliptical cylinder whose equation is $3x^2+y^2=5$, cut by the plane $z=7$, is an ellipse equal and parallel to the ellipse whose equation is $3x^2+y^2=5$.

143. Equations in three variables. Surfaces. A point is determined by three coördinates, which will vary according to some definite law if the point moves so as to trace some definite surface. Algebraically, the coördinates must then satisfy an equation having three variables: that is, *the locus of any algebraic equation in three variables is a surface.*

144. Curves. Projections of curves. Traces of surfaces. Two surfaces intersect in a curve in space; and since every algebraic equation in solid analytic geometry represents a surface, a curve may be represented analytically by the two equations, regarded as simultaneous, of surfaces which pass through it. Thus it has been seen that the equations $y=b$, $z=c$ separately represent planes, but considered as simultaneous they represent the straight line which is the intersection of those planes. But by the reasoning of Art. 38, the given equations of a curve may be replaced by simpler ones which represent other surfaces passing through the same curve. In dealing with curves it is often useful to obtain, from the equations given, equations of *cylinders* through the same curve; *i.e.*, it is generally useful to represent a curve by two equations, each in two variables only.

Since the cylinder is taken perpendicular to a coördinate plane, its equation when considered as of a plane curve, represents the **projection** of the space curve upon the coördinate plane.

Example. The curve of intersection of the two surfaces,

(1) $x^2+y^2+z^2-25=0$ and (2) $x^2+y^2-16=0$,

is also the intersection of the surface

$$x^2+y^2+z^2-25-(x^2+y^2-16)=0,\ i.e.,\ z=\pm 3, \quad (3)$$

with the surface (2). The curve is therefore composed of two circles of radius 4, parallel to the xy-plane at distances $+3$ and -3 from it.

Here equation (2) is itself the equation of the projection of the space curve upon the xy-plane; while $z=\pm 3$ gives the projection upon each of the other coördinate planes.

Similarly, the curves of intersection of a surface with the coördinate planes may be used to help determine the nature of a surface. These curves are called the **traces** of the surface. The **intercepts** upon the axes are of course also useful.

Thus, the trace of the surface $x^2+y^2+z^2=25$

on the yz-plane, is $y^2+z^2=25$;

on the zx-plane, is $x^2+z^2=25$;

on the xy-plane, is $x^2+y^2=25$.

Each of these traces is a circle of radius 5, about the origin as center; and the surface is a sphere of radius 5 with center at the origin.

Since three surfaces in general have only one or more separate points in common, the locus of three equations, considered as simultaneous, is one or more distinct points.

145. Surfaces of revolution. Analogous to the cylinders are the surfaces traced by revolving any plane curve about a straight line in its plane as axis. From the method of formation, it follows that each plane section perpendicular to the axis is a circle, — the path traced by a point of the generating curve as it revolves; and the radius of the circle is the distance of the point from the axis before revolution begins. These facts lead readily to the equation of any surface of revolution, as a few examples will show.

(*a*) *The cone formed by revolving about the z-axis the line*

$$2x+3z=15. \tag{1}$$

Any point P of the line (1) traces during the revolution a circle of radius LP, parallel to the xy-plane. The equation of that path is

$$x^2+y^2=\overline{LP}^2.$$

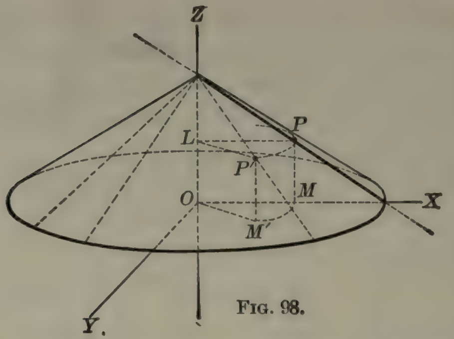

FIG. 98.

But in the xz-plane, before revolution is begun, LP is the abscissa of P and may be found in terms of the third coördinate, z, which does not change as P moves: by equation (1),

$$\overline{LP} = x = \frac{15 - 3z}{2};$$

so that the equation of the path of P is

$$x^2 + y^2 = \frac{(15 - 3z)^2}{4}. \tag{2}$$

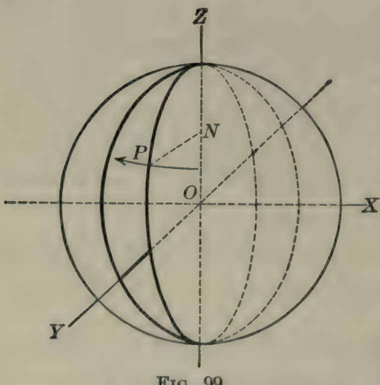

FIG. 99.

But P is *any* point of line (1); hence equation (2) is satisfied by every point of the line during its motion, *i.e.*, of the surface generated by the line; which is the required conical surface.

(*b*) *The sphere formed by revolving about the z-axis the circle*

$$x^2 + z^2 = 25. \tag{3}$$

In this case, any point P of the curve traces during the revolution a circle of radius NP, parallel to the xy-plane. The equation of this path is therefore

$$x^2 + y^2 = \overline{NP}^2.$$

But in the xz-plane, by equation (3),

$$NP = x = \sqrt{25 - z^2}.$$

Hence, substituting above,

$$x^2 + y^2 + z^2 = 25; \qquad (4)$$

which is the equation of the required spherical surface.

Of the various surfaces of revolution those of particular interest are generated by revolving about their axes the various conic sections, giving the cones, spheres, paraboloids, ellipsoids, and hyperboloids of revolution.

The student may verify the equations of the following surfaces: *

The sphere: with center at the origin, and radius r,

$$x^2 + y^2 + z^2 = r^2; \qquad (5)$$

with center at (a, b, c), by Art. 138, equation (5) becomes

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2. \qquad (6)$$

The cone: the surface generated by the right line $z = mx + c$, rotated about the z-axis,

$$x^2 + y^2 = \frac{(z-c)^2}{m^2}. \qquad (7)$$

The oblate spheroid: the surface generated by the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, rotated about the *minor* axis,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1. \qquad (8)$$

The prolate spheroid: the surface generated by the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, rotated about the *major* axis,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1. \qquad (9)$$

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* See Chapter XIII, where diagrams are given for the corresponding cases of the general quadric, with elliptical instead of circular sections.

The hyperboloid of one nappe: the surface generated by the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$, rotated about the *conjugate* axis,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=1. \tag{10}$$

The hyperboloid of two nappes: the surface generated by the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$, rotated about the *transverse* axis,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{b^2}=1. \tag{11}$$

The paraboloid of revolution: the surface generated by the parabola $x^2=4pz$, rotated about its axis,

$$x^2+y^2=4pz. \tag{12}$$

EXAMPLES ON CHAPTER XI

What is the locus of each of the following equations?

1. $25x^2-10x+1=0$.
2. $x^2+4x=0$.
3. $x^2-2xy+y^2+3x-3y=0$.
4. $ax^2=bxy-cy^2$.
5. $x^3+x^2-4x-4=0$.
6. $x^2+9y=9$.

What are the curves of intersection of the surfaces represented by the equations:

7. $x-1=0,\ x^2+y^2+z^2=10$?
8. $4x^2-9y^2=0,\ z=4$?
9. $x^2+y^2+z^2=9,\ 4x^2+y^2=4$?
10. $9(x^2+y^2)-z^2=25-10z,\ z=\pm 5$?
11. $3x^2-4y^2-z^2=12,\ \frac{x^2}{9}+\frac{y^2}{16}=1$?

Determine the traces upon the coördinate planes of the following surfaces:

12. $x^2+y^2=4z^2$;
13. $x^2-3y^2+2z^2=6$.

Find the equation of :

14. The paraboloid of revolution one of whose traces is $x^2 = 5z + 3$.

15. The cone of revolution one of whose traces is $z = y - 10$ and whose axis is the axis of y. Find its vertex.

16. The oblate spheroid one of whose traces is $\frac{z^2}{2} + \frac{x^2}{3} = 1$.

17. The prolate spheroid one of whose traces is

$$\frac{(x-1)^2}{7} + \frac{(z-1)^2}{9} = 1.$$

18. The surface of revolution whose axis is the x-axis and one of whose traces is $y^2x = -4$.

19. The hyperboloid of two nappes one of whose traces is $16y^2 - 9x^2 = 144$.

20. The sphere having the minor axis of the ellipse $4x^2 + 9y^2 - 6y - 4x = 0$ as diameter

CHAPTER XII

EQUATIONS OF THE FIRST DEGREE

$\boldsymbol{Ax + By + Cz + D = 0}$

PLANES AND STRAIGHT LINES

I. The Plane

146. Every equation of the first degree represents a plane. A plane is a surface such that it contains every point on a straight line joining any two of its points.

Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be any two points of the surface whose equation is

$$\boldsymbol{Ax + By + Cz + D = 0,} \qquad [72]$$

so that
$$Ax_1 + By_1 + Cz_1 + D = 0, \qquad (1)$$

and
$$Ax_2 + By_2 + Cz_2 + D = 0. \qquad (2)$$

Now, if $P_3 \equiv (x_3, y_3, z_3)$ be any point on the straight line from P_1 to P_2 at a distance d_1 from P_1 and d_2 from P_2, then, by equation [71],

$$x_3 = \frac{d_1x_2 + d_2x_1}{d_1 + d_2},\ y_3 = \frac{d_1y_2 + d_2y_1}{d_1 + d_2},\ z_3 = \frac{d_1z_2 + d_2z_1}{d_1 + d_2}. \qquad (3)$$

But this point lies on the surface represented by equation [72]; for, on substituting its coördinates in equation [72], the latter becomes

$$\frac{d_1}{d_1+d_2}(Ax_2 + By_2 + Cz_2 + D) + \frac{d_2}{d_1+d_2}(Ax_1 + By_1 + Cz_1 + D) = 0,$$

which is a true equation, since each number in parenthesis vanishes separately by equations (1) and (2). Hence every point of the line P_1P_2 is on the locus of equation [72], and that locus is therefore a plane. *Every algebraic equation of the first degree in three variables represents a plane.*

147. The intercept equation of a plane. A plane will in general cut each coördinate axis at some definite distance from the origin, and this distance is called the **intercept** of the plane on the axis. If a, b, c be the intercepts on the x-, y-, and z-axes, respectively, made by the plane whose equation is

$$Ax + By + Cz + D = 0, \tag{1}$$

then $(a, 0, 0,)$, $(0, b, 0)$, $(0, 0, c)$ are points of the plane, and therefore

$$Aa + D = 0, \quad Bb + D = 0, \quad Cc + D = 0,$$

i.e.,
$$A = -\frac{D}{a}, \quad B = -\frac{D}{b}, \quad C = -\frac{D}{c}. \tag{2}$$

Hence equation (1) may be written

$$\frac{Dx}{a} + \frac{Dy}{b} + \frac{Dz}{c} - D = 0,$$

i.e.,
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1; \tag{73}$$

and this is the equation of the plane in terms of its intercepts.

From equations (2) it is clear also that the intercepts of plane (1) are

$$a = -\frac{D}{A}, \quad b = -\frac{D}{B}, \quad c = -\frac{D}{C}. \tag{3}$$

148. The normal equation of a plane. A plane is wholly determined in position if the length and direction of a perpendicular upon it from the origin be known; and this method of fixing a plane leads to one of the most useful forms of its

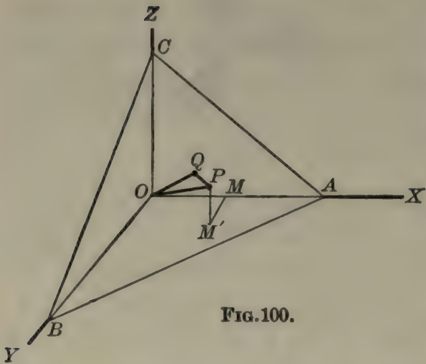

FIG. 100.

equation. Let OQ be the perpendicular from the origin O to the plane ABC, let p be its length, and let α, β, γ be its direction angles. Let $P \equiv (x, y, z)$ be any point of the plane (other than Q), and draw its coördinates OM, MM', and $M'P$.

Then, projecting upon OQ,

$$\text{proj. } OM + \text{proj. } MM' + \text{proj. } M'P = \text{proj. } OP,$$

that is,

$$\mathbf{x \cos \alpha + y \cos \beta + z \cos \gamma = p.} \qquad [74]$$

This is called the *normal* equation of the plane.

By reasoning analogous to that of Art. 46 it can be shown that the equation

$$Ax + By + Cz + D = 0,$$

when written in the normal form, becomes

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x + \frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B^2+C^2}}z = \frac{-D}{\sqrt{A^2+B^2+C^2}};$$

and therefore that

$$\left.\begin{aligned} \cos\alpha &= \frac{A}{\sqrt{A^2+B^2+C^2}}, & \cos\beta &= \frac{B}{\sqrt{A^2+B^2+C^2}}, \\ \cos\gamma &= \frac{C}{\sqrt{A^2+B^2+C^2}}, & p &= \frac{-D}{\sqrt{A^2+B^2+C^2}}. \end{aligned}\right\} \qquad [75]$$

Hence, to reduce equation (1) to the normal form, it is only necessary to *transpose the constant term to the second member*

of the equation, and then divide both members by the square root of the sum of the squares of the coefficients of the variable terms. The sign of the radical may be determined so that any chosen direction angle, as α, shall be less than 90°: then, *the sign of the radical is the same as the sign of the x-term,* i.e., in general, positive.

149. The angle between two planes. Parallel and perpendicular planes. The angles formed by two intersecting planes are the same as the angles formed by two straight lines perpendicular to them respectively; *i.e.*, are the same as the angles between the respective normals from the origin to the planes.

If $$A_1x + B_1y + C_1z + D_1 = 0, \tag{1}$$

and $$A_2x + B_2y + C_2z + D_2 = 0, \tag{2}$$

are two planes, then by equations [67] and [75], if θ be the angle between the two planes, and hence between the two normals,

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \qquad [76]$$

There are two cases of special interest.

(*a*) *Parallel planes.* If the planes (1) and (2) are parallel, their normals from the origin will have the same direction cosines, and differ only in length; therefore, by equations [75], the equations of the planes must be such that the coefficients of the variable terms are the same in the two equations, or can be made the same by multiplying one equation by a constant. In other words, if the planes (1) and (2) are parallel, then

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}; \qquad [77]$$

i.e., the plane $$Ax + By + Cz + K = 0 \tag{3}$$

is parallel to the plane

$$Ax + By + Cz + D = 0, \tag{4}$$

for all values of the parameter K.

(*b*) *Perpendicular planes.* If the planes (1) and (2) are perpendicular to each other, then $\cos\theta = 0$,

and $$\mathbf{A_1A_2 + B_1B_2 + C_1C_2 = 0;} \qquad [78]$$

and conversely.

II. The Straight Line

150. Two equations of the first degree represent a straight line. Every equation of first degree represents a plane, and two planes intersect in a straight line; hence the locus of the two simultaneous equations of first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,\ A_2x + B_2y + C_2z + D_2 = 0, \tag{1}$$

is a straight line. As suggested in Art. 144, it is generally more simple to represent the straight line by equations in two variables only, *standard forms*, to which equations (1) can always be reduced.

151. Standard forms for the equations of a straight line. (*a*) *The straight line through a given point in a given direction.* Let $P_1 \equiv (x_1, y_1, z_1)$ be a given point, and α, β, γ the direction angles of a straight line through it. Let $P \equiv (x, y, z)$ be any other point on the line, at a distance d from P_1. Then by equation [66],

$$d\cos\alpha = x - x_1,\quad d\cos\beta = y - y_1,\quad d\cos\gamma = z - z_1, \tag{1}$$

hence $$\frac{\mathbf{x - x_1}}{\mathbf{\cos\alpha}} = \frac{\mathbf{y - y_1}}{\mathbf{\cos\beta}} = \frac{\mathbf{z - z_1}}{\mathbf{\cos\gamma}}; \qquad [79]$$

which are the equations of a straight line in the first standard forms, called the *symmetrical equations.*

(*b*) *The straight line through two given points.* Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be the given points. Any straight line passing through P_1 has [79] for its equations. If the line passes also through P_2, then

$$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}; \tag{2}$$

and hence, dividing equations [79] by equations (2), to eliminate the unknown direction cosines,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad [80]$$

These are the second standard forms for the equations of a straight line.

(*c*) *The straight line with given projections on the coördinate planes.* If the equation of the projections of a given line upon the *zx*- and *yz*-planes are, respectively,

$$\left.\begin{aligned} x &= mz + b, \\ y &= nz + d, \end{aligned}\right\} \qquad [81]$$

then, considered as simultaneous, these also are the equations of the given line in space.

152. Reduction of the general equations of a straight line to a standard form. Determination of the direction angles.

(*a*) *Third standard form: projections.* This reduction may be illustrated by a numerical example.

Given the equations

$$3x + 2y + z - 5 = 0, \quad x + 2y - 2z = 3, \tag{1}$$

representing a straight line. Eliminating z, y, and x, successively, the equations

$$7x + 6y - 13 = 0, \; 2x + 3z - 2 = 0, \; 4y - 7z - 4 = 0 \tag{2}$$

are obtained, each representing a plane through the given line

and perpendicular to a coördinate plane. Therefore these equations are also the equations of the projections of the line, upon the xy-, zx-, and yz-planes, respectively. Any two of them will represent the line

(*b*) *First standard form: direction angles.* The method of reducing the general equations of a straight line to the first standard form, and finding its direction angles, can also be illustrated by a numerical case.

Considering still the line whose equations are (1) above and taking the equations of any two of its projections,

e.g., $$2x+3z-2=0,\ 4y-7z-4=0; \tag{3}$$

these have one variable, z, in common. Equating the values of this common variable from the two equations, gives

$$z=\frac{-2x+2}{3}=\frac{4y-4}{7},$$

which may be written, to correspond with equations [79],

$$\frac{x-1}{-\frac{3}{2}}=\frac{y-1}{\frac{7}{4}}=\frac{z-0}{1}. \tag{4}$$

Now, although the denominators $-\frac{3}{2}, \frac{7}{4}, 1$ of equations (4) are not direction cosines of any line, yet, by equations [64], they differ from such direction cosines only by the divisor

$$\sqrt{\tfrac{9}{4}+\tfrac{49}{16}+1},\ i.e., \text{ by } \tfrac{1}{4}\sqrt{101}.$$

or, rewriting equations (4) in the form

$$\frac{x-1}{\frac{-6}{\sqrt{101}}}=\frac{y-1}{\frac{7}{\sqrt{101}}}=\frac{z-0}{\frac{4}{\sqrt{101}}}, \tag{5}$$

they correspond entirely to equations [79]. Therefore the line passes through the point (1, 1, 0), and its direction angles are given by the relations

$$\cos\alpha = -\frac{6}{\sqrt{101}},\ \cos\beta = \frac{7}{\sqrt{101}},\ \cos\gamma = \frac{4}{\sqrt{101}}.$$

The method given above is evidently entirely general.

EXAMPLES ON CHAPTER XII

1. Find the equations of a line through the points $(^-1, {}^-2, {}^-3)$ and $(3, 0, 1)$.

2. Find the equation of a plane through the points $(0, 0, 0)$, $(0, 1, {}^-1)$, and $(5, 4, 3)$.

3. Write the equations of the straight line through the point (a, b, c), and having its direction cosines proportional to l, m, n.

4. What are the projections of the line of Ex. 1 upon the coördinate planes? Where does the line pierce those planes?

5. Find the equations of a straight line through the point $(^-1, 2, {}^-3)$ and perpendicular to the plane $x + 3y + 3z = 0$.

Reduce to the intercept and normal forms, and determine which octant each plane cuts:

6. $2x + 3y + 10z = 7$; **7.** $ay + bz - 10 = cx$.

8. Reduce the equations of the line

$$2x - 3y - z = 7, \quad 5y + 2z - 1 = x$$

to the symmetrical form, and determine its direction cosines.

9. What does equation [74] become if the plane is perpendicular to the xy-plane? What if parallel to the xy-plane?

10. Derive the formula for the distance of the point $P_1 \equiv (x_1, y_1, z_1)$ from the plane $Ax + By + Cz + D = 0$.

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}. \qquad [82]$$

HINT: Pass a parallel plane through P_1, and find the distance p for each plane. [Cf. Art. 51.]

11. Write the equation of a plane parallel to the plane

$$3x - 4y + 10z - 10 = 0,$$

and passing through the origin; through the point $(^-4, 5, 6)$.

12. Write the equation of a plane perpendicular to the plane

$$3x + 5y - z + 6 = 0,$$

and passing through the two points (3, 1, 2) and $(0, ^-2, ^-4)$.

13. Find the distances of the points $(^-7, ^+2, ^-3)$ and $(^-3, ^-3, 1)$ from the plane $2x + 5y - z - 9 = 0$. Are they on the same side of the plane? [See Ex. 10].

14. At what angle does the plane $ax + by + cz = 0$ cut each coördinate plane? Each coördinate axis?

15. Find the equation of a plane through the point (1, 1, 1) and perpendicular to each of the planes

$$2x - 3y + 7z = 1, \quad x - y - 2z = 2.$$

16. Write the equation of a plane at the distance d from the origin and inclined equally to the three coördinate planes.

17. Write the equations of a straight line through the point $(^-4, ^-5, ^-6)$ and parallel to the line

$$2x - 3z + y = 0, \quad x + y + z = 0.$$

18. Find the projections on the coördinate planes of the line

$$3x - 3z + y - 2 = 0, \quad x + y + z + 1 = 0.$$

19. Prove that the planes

$$x - 4y = 0, \quad y - 5 = 0, \quad x - 20 = 0$$

intersect in a line. Find its equations.

CHAPTER XIII

EQUATIONS OF THE SECOND DEGREE

QUADRIC SURFACES

153. The locus of an equation of second degree. The most general algebraic equation of second degree in three variables may be written

$$\begin{aligned}&\boldsymbol{Ax^2 + By^2 + Cz^2 + 2\,Fyz + 2\,Gxz + 2\,Hxy + 2\,Lx + 2\,My}\\&\quad\boldsymbol{+ 2\,Nz + K = 0.}\end{aligned} \qquad \textbf{[83]}$$

Any surface which is the locus of an equation of second degree is called a **quadric** surface, and is of particular interest because of its close connection with and analogy to the conic sections. In fact, every plane section of a quadric is a conic, as may be easily shown as follows.

By Art. 139, Exs. 13, 14, any plane may be chosen as a coördinate plane, and the transformation of coördinates to the new axes — the equations of transformation being in each case of the first degree — will leave the degree of equation [83] unchanged;* *i.e.*, the new equation of the locus will still be the form [83], though with different values for the coefficients. To find the nature of any plane section, choose the given plane as (say) the xy-plane of reference, and transform to the

* Although in this book rectangular axes are used, the same reasoning holds for oblique coördinates. [Cf. Art. 35.]

new axes; the new equation will be of form [83]. Then let $z=0$. The equation of the section of the quadric is

$$Ax^2+By^2+2Hxy+2Lx+2My+K=0; \qquad (1)$$

and this, by Art. 98, represents a conic.

Moreover, the trace of the surface on any parallel plane, as $z=a$, is given by the equation

$$Ax^2+By^2+2Hxy+2(L+aG)x+2(M+aF)y$$
$$+(Ca^2+2Na+K)=0. \qquad (2)$$

Now, by Art. 101, the loci of equations (1) and (2) are conics of the same species, and with semi-axes proportional; therefore their eccentricities are equal, and the curves are similar. Hence, *all parallel plane sections of a quadric are similar conics.*

154. Species of quadrics. Simplified equation of second degree. Quadric surfaces may be conveniently classed under four species. For, although different plane sections of any surface will in general be conics of different species, still the *general* form of the surface may be characterized most strikingly by those plane sections which are ellipses, hyperbolas, parabolas, or straight lines. These species are called, respectively, *ellipsoids, hyperboloids, paraboloids,* and *cones;* and each species has special varieties, depending upon the nature of a second system of plane sections.

155. The ellipsoid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. From the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad [84]$$

the following properties of its locus may be derived:

(1) The traces on each coördinate plane are ellipses, having the semi-axes a and b in the xy-plane, b and c in the yz-plane, and c and a in the zx-plane.

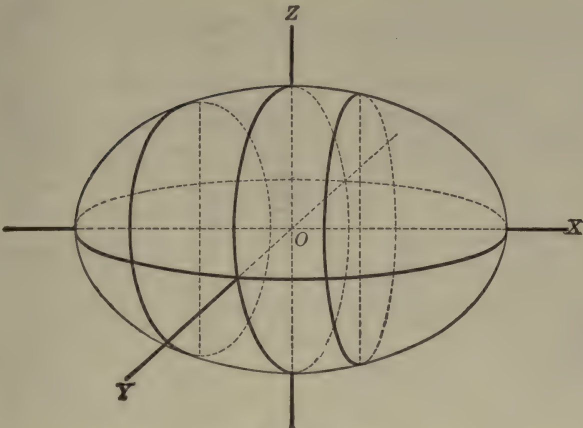

FIG. 101.

(2) The traces on planes parallel to any coördinate plane are similar ellipses [Art. 153].

(3) The equation may be written

$$\frac{y^2}{\frac{b^2(a^2-x^2)}{a^2}}+\frac{z^2}{\frac{c^2(a^2-x^2)}{a^2}}=1;$$

hence for a plane section parallel to the yz-plane, the semi-axes are real if the value of x lies between $-a$ and $+a$, imaginary if beyond those limits, and zero if $x=\pm a$. Moreover, the lengths of the axes diminish continuously from the values b and c, respectively, when $x=0$ to the value zero when $x=\pm a$.

Similarly for sections parallel to either of the other coördinate planes.

(4) The surface is symmetrical with respect to each coördinate plane.

This quadric surface, the locus of equation [84], is called an **ellipsoid**. It may be conceived as generated by a variable

ellipse, which has its vertices upon, and moves always perpendicular to, two fixed ellipses which in turn are perpendicular to each other and have one axis in common.

From this definition equation [84] can be easily derived. Let CRA and ASB be fixed ellipses perpendicular to each other, and having the semi-axis OA in common, and the second axes OC and OB, respectively; and let SPR be the variable ellipse, with semi-axes MS and MR. If OA, OB, OC be taken as the x, y, and z axes, respectively; and P be any point on the moving ellipse with coördinates OM, MM', $M'P$, then [by Art. 88],

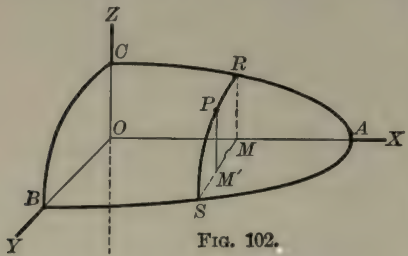

FIG. 102.

$$\frac{\overline{M'P}^2}{\overline{MR}^2}+\frac{\overline{MM'}^2}{\overline{MS}^2}=1, \quad \frac{\overline{MR}^2}{\overline{OC}^2}+\frac{\overline{OM}^2}{\overline{OA}^2}=1, \quad \frac{\overline{MS}^2}{\overline{OB}^2}+\frac{\overline{OM}^2}{\overline{OA}^2}=1,$$

$$i.e., \frac{z^2}{\overline{MR}^2}+\frac{y^2}{\overline{MS}^2}=1, \ (1) \quad \frac{\overline{MR}^2}{c^2}+\frac{x^2}{a^2}=1, \ (2) \quad \frac{\overline{MS}^2}{b^2}+\frac{x^2}{a^2}=1. \ (3)$$

By equations (2) and (3),

$$\overline{MR}^2=c^2\left(1-\frac{x^2}{a^2}\right), \quad \overline{MS}^2=b^2\left(1-\frac{x^2}{a^2}\right).$$

Substitution in (1) gives $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Moreover, every algebraic equation of the form

$$Ax^2+By^2+Cz^2-K=0$$

represents an ellipsoid. If two of the coefficients of the variable terms are equal, it is an ellipsoid of revolution, either an oblate or prolate spheriod; and if the three coefficients of the

variable terms are equal, it is a sphere. [Cf. Art. 145, equations (8), (9), and (6)].

156. The un-parted hyperboloid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$
From the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{85}$$

the following properties of its locus may be derived:

(1) The trace on the xy-plane is an ellipse, with semi-axes a and b; while the traces on the yz- and zx-planes are hyperbolas, having the semi-axes b and c, and c and a, respectively, with the conjugate axes along the z-axis.

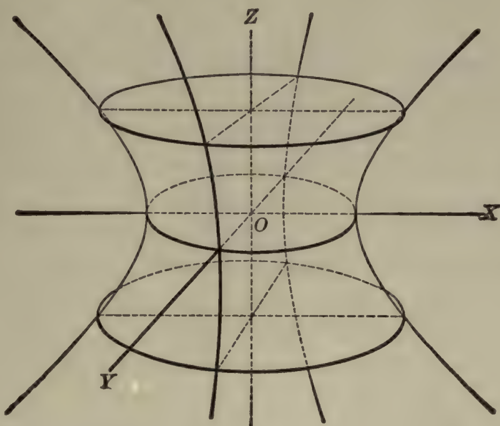

FIG. 103.

(2) The traces on planes parallel to any coördinate plane are similar conics, ellipses, or hyperbolas, respectively [Art. 153].

(3) The traces on the planes $x=a$, $x=-a$, $y=b$, $y=-b$ are in each case a pair of intersecting straight lines.

(4) The equation may be written

$$\frac{x^2}{\frac{a^2(c^2+z^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2+z^2)}{c^2}} = 1, \tag{1}$$

also

$$\frac{y^2}{\frac{b^2(a^2-x^2)}{a^2}} - \frac{z^2}{\frac{c^2(a^2-x^2)}{a^2}} = 1. \tag{2}$$

From equation (1) it appears that the trace on the xy-plane is the smallest of the system of ellipses parallel to that plane, and that the sections increase continuously and indefinitely as z increases from 0 to $\pm\infty$.

From equation (2) it appears that the tranverse axis of the hyperbolas parallel to the yz-plane is parallel to the y-axis. Similarly, for the xz-sections the transverse axis is parallel to the x-axis.

(5) The surface is symmetrical with respect to each coordinate plane.

This quadric surface, whose equation is [85], is called an **un-parted hyperboloid**, or an **hyperboloid of one sheet.** It may be conceived as generated by a variable ellipse, which has its vertices upon and moves always perpendicular to two fixed hyperbolas, which in turn are perpendicular to each other, and have a common conjugate axis. Its equation can be readily obtained from this definition.*

Moreover, every equation of the form $Ax^2+By^2-Cz^2-K=0$ represents an un-parted hyperboloid. If the two positive coefficients are equal, *i.e.*, if $a=b$, the quadric is the simple hyperboloid of revolution [Art. 145, equation (10)].

157. The bi-parted hyperboloid: equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$ From the equation

* Cf. Art. 155.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad [86]$$

the following properties of its locus may be derived:

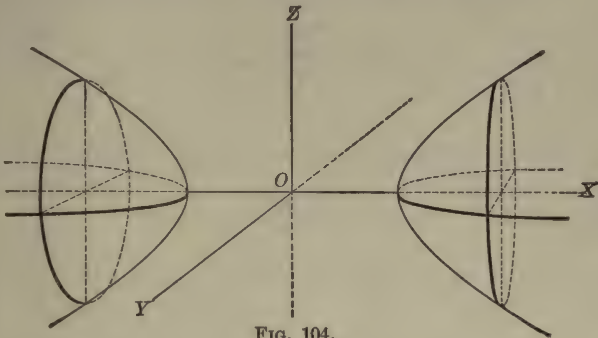

FIG. 104.

(1) The traces on the *xy*- and *zx*-planes are hyperbolas, having semi-axes a and b, and c and a, respectively, with the transverse axis along the x-axis; while the traces on the planes parallel to the *yz*-plane are imaginary if x lies between a and $-a$, real ellipses if x is beyond those limits, and points if $x = \pm a$.

(2) The traces on planes parallel to any coördinate plane are similar [Art 153].

(3) The elliptical sections parallel to the *yz*-plane increase continuously and indefinitely as x varies from $+a$ to $+\infty$, or from $-a$ to $-\infty$.

(4) The surface is symmetrical with respect to each coördinate plane.

This quadric surface, whose equation is [86], is called a **bi-parted hyperboloid** or **hyperboloid of two sheets.** It may be conceived as generated by a variable ellipse which has its vertices upon, and moves always perpendicular to, two fixed hyperbolas which in turn are perpendicular to each other, and

have a common transverse axis. This definition leads readily to the equation [86].*

Moreover, every equation of the form $Ax^2 - By^2 - Cz^2 - K = 0$ represents a bi-parted hyperboloid. If the coefficients of the two negative variable terms are equal, *i.e.*, if $b = c$, the surface is the (double) hyperboloid of revolution. [Cf. Art. 145, equation (11)].

158. The paraboloids: equation $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = z$. A discussion of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \qquad [87]$$

similar to that of the preceding articles shows that its locus is as represented in Fig. 105; it is symmetrical with respect to the yz- and zx-plane, but not with respect to the xy-plane. This quadric is the **elliptic paraboloid**, and may be conceived as being generated by a variable parabola which has its vertex upon, and moves always perpendicular to, a fixed parabola, the axes of the two parabolas being parallel and lying in the *same* direction. This definition leads directly to equation [87].*

FIG 105.

Every equation of the form $Ax^2 + By^2 - 2\,Nz = 0$ represents an elliptic paraboloid. If the two positive coefficients are equal, the quadric is a paraboloid of revolution. [Cf. Art. 145, equation (12)].

Similarly, the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$ [88]

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* See Art. 155.

has for its locus a surface such as is represented in Fig. 106. This quadric is the **hyperbolic paraboloid**, and may be conceived as generated by a variable parabola which has its vertex upon

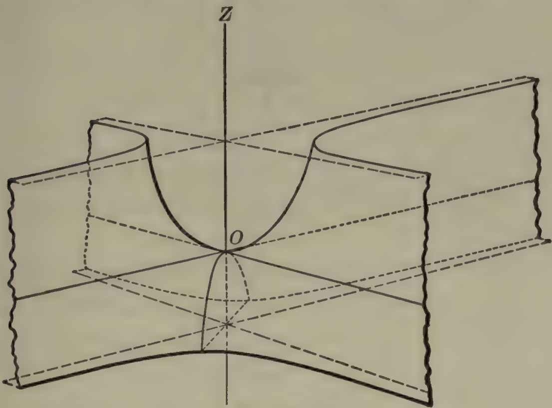

FIG. 106.

and moves always perpendicular to a fixed parabola, the axes of the two parabolas being parallel, but lying in *opposite* directions. Equation [88] may be derived at once from this definition.*

Every equation of the form $Ax^2 - By^2 - 2Nz = 0$ represents an hyperbolic paraboloid.

159. The cone: equation $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0}$. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ evidently is satisfied by the coördinates of only one real point, *viz.*, the origin. But the equation

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0} \qquad [89]$$

has a locus of importance; it has the following properties:

* See Art. 155.

(1) The origin is a point of the locus.

(2) The trace on the xy-plane is a point. The traces on planes parallel to the xy-plane are similar ellipses, whose semi-axes increase continuously and indefinitely as z increases from 0 to $\pm\infty$.

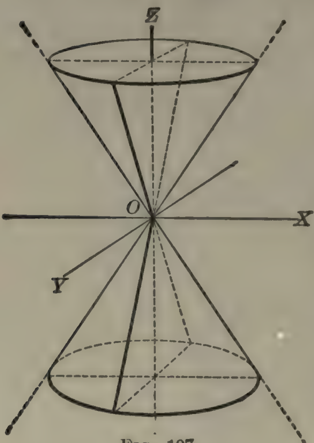

FIG. 107.

(3) The trace on each of the other coördinate planes is a pair of straight lines which intersect at the origin.

(4) The surface is symmetrical with respect to each coördinate plane, hence also with respect to the origin.

(5) The straight line through the origin and any other point of the locus lies wholly in the locus.

This quadric surface is called a **cone**, and the origin is its **vertex**. It may be conceived as generated by a straight line which moves along a fixed ellipse as directrix, and passes through a fixed point in a straight line which is perpendicular to the plane of the ellipse at its center.

Every equation of the form $Ax^2 + By^2 - Cz^2 = 0$ represents a cone. If the two positive coefficients are equal, it is a cone of revolution, or circular cone. [Cf. Art. 145, equation (7)].

The reasoning of Art. 153, applied to the special equation of the form [83] which represents a cone, gives an analytic proof of the fact that every plane section of a cone is a second degree curve. [Cf. Art. 77].

160. Ruled surfaces. Equation [85], of the un-parted hyperboloid, may be written

$$\left(\frac{y}{b}-\frac{z}{c}\right)\left(\frac{y}{b}+\frac{z}{c}\right)=\left(1-\frac{x}{a}\right)\left(1+\frac{x}{a}\right). \tag{1}$$

If now the equations
$$\frac{y}{b}-\frac{z}{c}=k\left(1-\frac{x}{a}\right) \tag{2}$$
and
$$k\left(\frac{y}{b}+\frac{z}{c}\right)=1+\frac{x}{a} \tag{3}$$
be formed, wherein k is any constant, each will represent a plane; and the two planes will intersect in a straight line, which may readily be shown to lie on the given surface. By varying k a family of such straight lines will be obtained.

The unparted hyperboloid is an example of a **ruled surface,** *i.e.*, of a surface such that through every point of it a straight line can be drawn which will be wholly on the surface.* The hyperbolic paraboloid and the cone are further examples of ruled surfaces, as may be shown by treating their equations like the above.

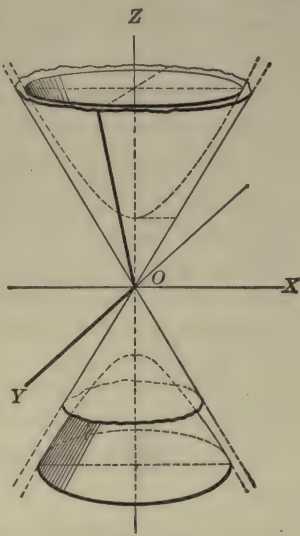

FIG. 108

161. The hyperboloid and its asymptotic cone. The hyperboloid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$
and the cone
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$
are closely related. It is clear that, since the equations differ only in the constant terms, the surfaces can have no finite points in common; while as the values of y and z are increased indefinitely, the corresponding values for x from the two equations become

* See Salmon's "Geometry of Three Dimensions," Chap. VI.

relatively nearer. In fact, the hyperboloid may be said to be tangent to the cone at infinity, and bears to the cone a relation entirely analogous to that between the hyperbola and its asymptotes. In the same way, the cone $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ is asymptotic to the hyperboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c}=1$.

EXAMPLES ON CHAPTER XIII

1. Derive Eq. [85] directly from the definition of Art. 156.

2. Derive Eq. [86] directly from the definition of Art. 157

3. Derive Eq. [89] directly from the definition of Art. 159.

4. Show analytically that the intersection of two spheres is a circle.

5. Find the equation of the tangent plane to the sphere $(x-a)^2+(y-b)^2+(z-c)^2=r^2$, at any point of the sphere.

6. Show that the equation $Ax_1x+By_1y+Cz_1z+K=0$ represents a plane tangent to the quadric, $Ax^2+By^2+Cz^2+K=0$, at the point (x_1, y_1, z_1) on the quadric.

7. Find the equation of the cone with origin as vertex and the ellipse $\frac{x^2}{25}+\frac{z^2}{9}=1$ in the plane $y=2$, as directrix.

8. Find the equation of a sphere having the line from $P_1\equiv(x_1, y_1, z_1)$ to $P_2\equiv(x_2, y_2, z_2)$ as a diameter.

9. Show that a sphere is determined by four points in space.

10. Write the equation of the quadric whose directing curves have the equations:

$$\frac{x^2}{64}+\frac{y^2}{4}=1, \text{ and } \frac{y^2}{4}+\frac{z^2}{64}=1.$$

CHAPTER XIV

CURVE TRACING. HIGHER PLANE CURVES*

162. Definitions. In Cartesian coördinates a curve whose equation is reducible to a finite number of terms, each involving only positive integral powers of the coördinates, is called an **algebraic** curve; all other curves are called **transcendental.**

Algebraic curves the degree of whose equations exceeds two, and all transcendental curves that lie wholly in a plane, are called **higher plane curves.***

In plotting plane curves, as suggested in Art. 30, the "point-wise" method is often puzzling, as well as long and tedious. It is usually helpful to find the intercepts and a few of the most easily obtained points; but then application should be made of the symmetry and other peculiarities of the equation. In this way tangents, diameters, and asymptotes often can be found, which may aid very effectively in determining the general shape of the locus.

Among the following curves are given many that are usually studied in more detail by calculus methods, including some of special historic interest.

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* The material in the following pages is intended merely to give brief suggestions for further drill in curve tracing. For a fuller treatment of Higher Plane Curves, with historical notes, see Tanner and Allen's "Elementary Course in Analytic Geometry."

163. Illustrative examples. *Diameters and tangents.* If the given equation is of the second degree in one of its variables, a diameter and tangents of the locus may sometimes be obtained, and the curve traced approximately, by the method illustrated below.

(*a*) *The conic* $2x^2 + 6xy + 4y^2 - 8x - 13y + 12 = 0.$ (1)

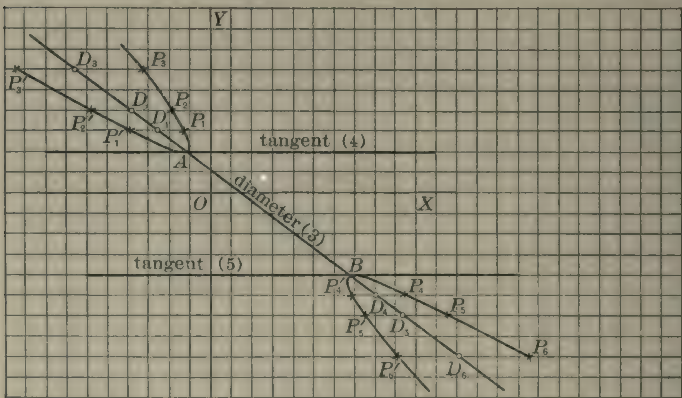

Fig. 109.

Solve for x:

$$x = -\tfrac{3}{2}y + 2 \pm \tfrac{1}{2}\sqrt{(y+4)(y-2)}. \qquad (2)$$

Plot the straight line:

$$x = -\tfrac{3}{2}y + 2: \qquad (3)$$

this line is a **diameter** of the locus. For, given any ordinate, say $y = 4$, then for the point D_2 on line (3) the abscissa is $x_d = -4$, while for the the corresponding points P_2 and P_2' on the locus (2),

$$x_2 = -4 + 2 = -2,$$

and $$x_2' = -4 - 2 = -6;$$

so that the chord $P_2 P_2'$ is bisected by the point D_2. Similarly, for any ordinate y_1, the point D_1, of the line (3) bisects the corresponding chord $P_1' P_1$ of the locus (2); hence, by Art. 108, line (3) is a **diameter**.

The length of the chord is in each case equal to the radical,

$$\sqrt{(y_1+4)(y_1-2)};$$

e.g., $$P_2'P_2=4, \text{ and } P_6'P_6=2\sqrt{10}.$$

Again, plot the lines:

$$y-2=0, \tag{4}$$

and $$y+4=0, \tag{5}$$

suggested by the factors in the radical. Now if $y_1=2$, one point, only, of the curve is found, *viz.*, A, which lies also in line (4); hence, line (4) is a **tangent** to the locus. Similarly, line (5) is a tangent.

If y_1 be taken between $+2$ and -4 the radical becomes imaginary. Hence, no points of the locus lie between these tangents. Numerically, x increases indefinitely as y increases beyond -4 in the negative direction, or beyond $+2$ in the positive direction.

Finally, particular points can be found by calculating $R=\frac{1}{2}\sqrt{(y+4)(y-2)}$ and plotting on either side of the corresponding point D of the diameter, as in the figure.

y	R	P
3	$\frac{1}{2}\sqrt{7} = \pm 1.3$	P_1
4	$\frac{1}{2}\sqrt{16} = \pm 2.0$	P_2
6	$\sqrt{10} = \pm 3.2$	P_3
-5	± 1.3	P_4
-6	± 2.0	P_5
-8	± 3.2	P_6

(*b*) *The curve* $$y^2=\frac{x^3}{2a-x}.$$

From the equation it follows that:

(1) The origin is a point of the curve (because there is no constant term).

(2) The curve is symmetrical with respect to the x-axis, only (because the y-term is of even degree).

(3) It lies wholly in the first and fourth quadrants (because the negative values of x give imaginary values for y).

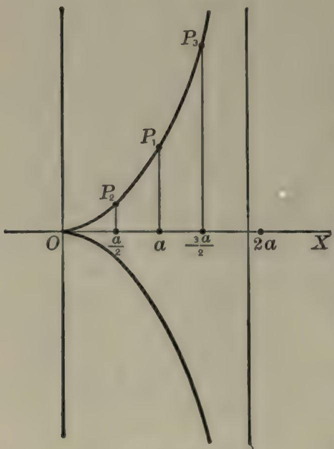

FIG. 110.

(4) It lies wholly to the left of the line $x - 2a = 0$, but approaches this line more and more closely as y increases (because if $x > 2a$, then y is imaginary, and as $x \doteq 2a$, y becomes infinite).

(5) The points $(a, \pm a)$,

$$\left(\frac{a}{2}, \pm 0.29\, a\right), \quad \left(\frac{3\,a}{2}, \pm 2.6\, a\right),$$

etc., are on the curve.

This curve is represented in figure 110. It is the **Cissoid of Diocles,** and was invented for the purpose of solving the famous problem of the "duplication of the cube."

(c) *The curve* $\qquad y = \dfrac{8\,a^3}{x^2 + 4\,a^2}.$

(1) The curve lies wholly in the first and second quadrants.

(2) It is symmetrical with respect to the y-axis, only.

(3) If $x = 0$, $y = 2\,a$, which is the maximum value for y.

(4) As x increases indefinitely, the curve approaches the x-axis.

(5) The points $(\pm a, 1.6\,a)$, $(\pm 2\,a, a)$, $(3\,a, .6\,a)$, etc., are on the curve. The curve is shown in figure 111, tangent to the line $y - 2\,a = 0$. It is called the **Witch of Agnesi.**

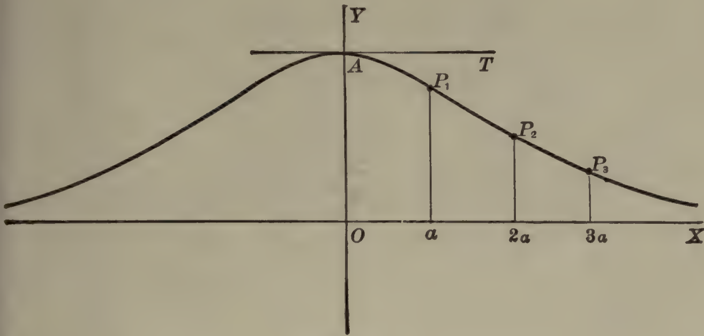

FIG. 111.

EXERCISES

Trace approximately the loci of the following equations:

1. $2x^2 + y^2 - 4xy + 19x - 6y + 3 = 0.$

2. $x^2 + 2y^2 - 2xy - x - 2y + 2 = 0.$

3. $x^2 + y^2 + 2xy + x - y + 1 = 0.$

4. $y = \dfrac{6}{x-3}.$ 5. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$ 6. $y = \dfrac{x+3}{x-3}.$

Show in detail the nature of the loci of the following equations:

7. $a^2y = x^3$;
the cubical parabola.

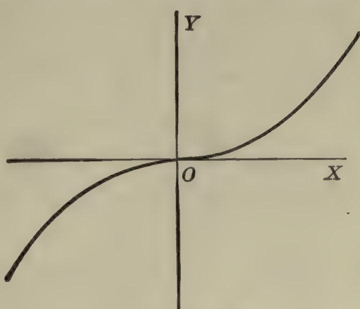

FIG. 112.

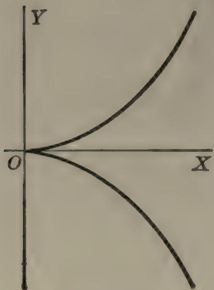

FIG. 113.

8. $ay^2 = x^3$: the semicubical parabola.

9. $a^4y^2 = a^2x^4 - x^6$.

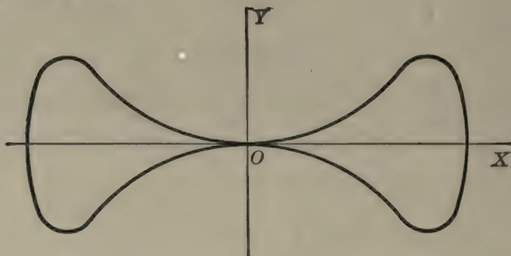

FIG. 114.

164. Asymptotes. As has been seen with the hyperbola, asymptotes, if any exist, are of decided help in tracing curves. Their equations may by found by a method similar to that of Art. 130, making use of the following algebraic theorem.

If in an equation of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + p = 0$$

the coefficients of the two highest powers of x approach zero as a limit, then two of the roots become infinite.*

---

* For, let $y = \frac{1}{x}$ so that y is a root of the equation

$$a + by + cy^2 + dy^3 + \cdots + py^n = 0.$$

The method may be illustrated by an example:

Given the hyperbola

$$xy - 3y - x - 3 = 0;$$

let

$$y = mx + c$$

be the equation of an asymptote, if one exists; then these two curves have two points of intersection at infinity.

Eliminating y, $mx^2 + (c - 3m - 1)x - 3c - 3 = 0.$

Two roots of this equation are infinite;

hence $\quad m = 0$ and $c - 3m - 1 = 0$: *i.e.*, $c = 1.$

Therefore, $\quad y = 0 \cdot x + 1$, *i.e.*, $y = 1$

is the equation of an asymptote.

There is one further possibility, viz., that there is an asymptote whose equation is not included in the form $y = mx + c$, because the line is perpendicular to the x-axis. [Art. 45].

In this case, $x = k$, where k is constant, and the corresponding ordinate y is infinite.

Now if $x = k$, then $y = \dfrac{k+3}{k-3}$;

and y approaches infinity as k approaches 3.

Hence

$$x = 3$$

is the equation of an asymptote.

The hyperbola can now be traced, as in Art. 163, Ex. 6.

Now if a and b approach zero, this equation approaches the form

$$y^2(c + dy + \cdots + py^{n-2}) = 0,$$

which has two roots equal to zero. But as y approaches zero, the corresponding value of x becomes infinite. That is, as a and b approach zero, two of the x-roots become infinite.

EXERCISES

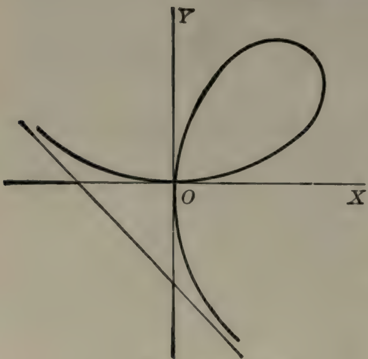

FIG. 115.

1. Show that $x + y + a = 0$, is an asymptote to the curve $x^3 + y^3 - 3\,axy = 0$, and trace the curve.

This curve is the **Folium of Descartes.**

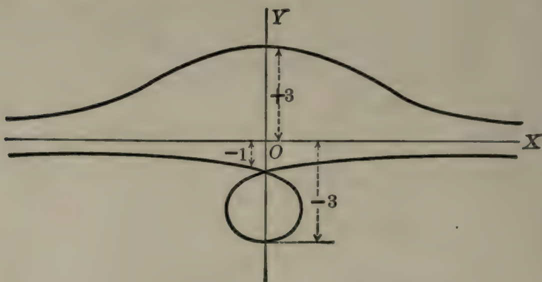

FIG. 116.

2. Trace the curve $x^2y^2 = (y + 1)^2(9 - y^2)$.

This curve is an example of the **Conchoid of Nicomedes.** The conchoid may be used to solve another famous problem of the ancients, the "trisection of an angle," as well as to accomplish the "duplication of the cube."

3. Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; **the hypocycloid.**

4. Trace the curve $x^2 + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

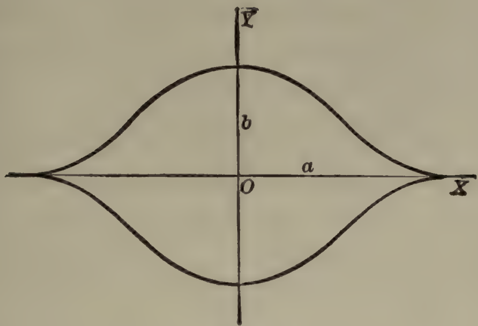

FIG. 117.

5. Trace the curve $$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$$

165. Exponential and logarithmic curves. An **exponential curve** has an equation of the form

$$x=a^y.$$

Since this equation may be written

$$y=\log_a x,$$

the locus may also be called a **logarithmic curve.**

Trace $$y=\log_{10} x.$$

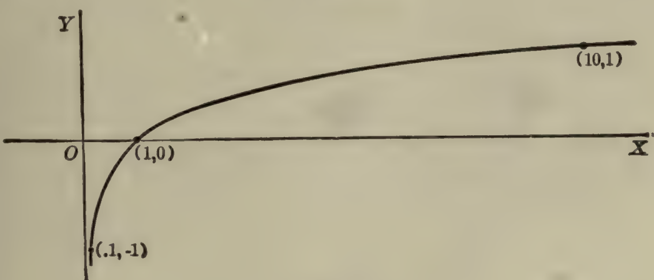

FIG. 118.

(1) The curve lies wholly in the first and fourth quadrants (since there are no logarithms of negative numbers).

(2) The line $x=0$ is an asymptote.

(3) As x increases, y increases, indefinitely.

(4) The points (1, 0), (10, 1), (100, 2), (0.1, -1) are on the curve.

EXERCISES

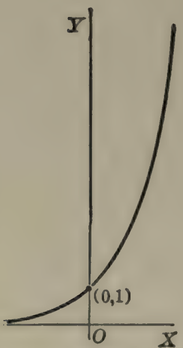

FIG. 119.

1. Trace $y=\log_2 x.$

Use the fact that

$$\log_2 x=\frac{1}{\log_{10} 2}\log_{10} x=\frac{1}{0.301}\log_{10} x$$
$$=3.32\log_{10} x.$$

2. Trace

$$y=\log_e x \text{ (where } e=2.71828).$$

$$\log_e x=\frac{1}{\log_{10} e}\log_{10} x=2.302\log_{10} x.$$

3. Trace $y=e^x.$

166. Trigonometric curves. Trigonometric curves may be plotted by points, using trigonometric tables; or they may be found by using line functions, from a circle with radius 1.

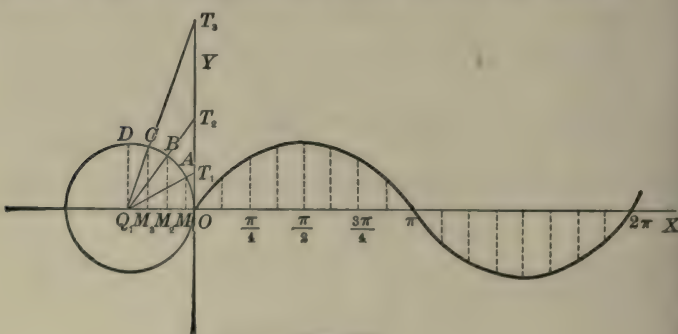

FIG. 120.

E.g., trace $y = \sin x.$

Draw a circle tangent to the y-axis on the negative side, with its center Q on the x-axis, and radius 1. Lay off from the origin on the x-axis the distances $2\pi = 6.28$, π, $\frac{\pi}{2}$, etc., and at the respective points of division erect perpendiculars.

In the circle lay off central angles of $\frac{\pi}{2}$ radians, $\frac{\pi}{4}$ radians, $\frac{\pi}{8}$ radians, etc., thus getting points D, B, A, etc., from which let fall perpendiculars upon OX, *viz.*, AM_1, BM_2, CM_3, etc. These lines represent the sines of the corresponding angles.

Therefore, for $x = \frac{\pi}{8}$, $y = M_1A$; for $x = \frac{\pi}{4}$, $y = M_2B$, etc.; and similarly, for $x = \frac{9\pi}{8}$, $y = -M_1A$; for $x = \frac{5\pi}{4}$, $y = -M_2B$, and so on.

The curve therefore consists of an infinite series of waves, as in the figure. The largest value of y is called the **amplitude of the wave.**

EXERCISES

Trace the loci:

1. $y = \cos x.$ **2.** $y = \tan x.$ **3.** $y = \sec x.$

4. $y = \cot x.$ **5.** $x = \sin y.$ **6.** $y = \tan^{-1} x.$

167. Loci of combined equations. If two curves are plotted on the same axes, and to the same scale, a new curve can be obtained by adding the ordinates of those points of the curves which have a given abscissa, to get the ordinate of the corresponding point of the new curve, with the same abscissa.

For example: plotting

$$(1)\quad y = e^x \text{ and } (2)\quad y = e^{-x}$$

then adding corresponding ordinates, the curve

$$y = e^x + e^{-x}$$

is obtained. This curve is called a **catenary**, and is the curve formed approximately by a cord, or chain, suspended freely between two points.

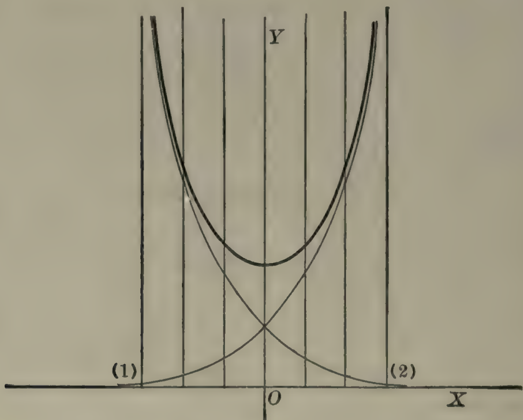

FIG. 121.

EXERCISES

Plot the curves:

1. $y = \sin x + \cos x$.

2. $y = 2x + \sin x$.

3. $y = e^x - \sin 2x$.

4. $y = x^2 + 2x - 3$.

5. Show by the method of Art. 167 how to solve graphically the equation

$$x^2 + (2x - 3) = 0.$$

Solve graphically the equations:

6. $3x^2 - 5x + 2 = 0$.

7. $\text{Sin } 2x = 2\cos x$.

168. Polar coördinates. A plane curve in polar coördinates is traced by methods similar to those used with Cartesian coördinates. A few examples will illustrate the method.

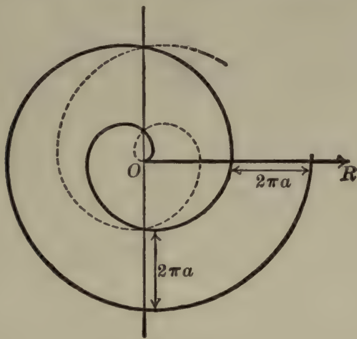

FIG. 122.

(a) *Trace* $$r = a\theta.$$

The angle θ must, of course, be measured in radians, so that r and θ may be plotted in units of length.

(1) r increases indefinitely as θ increases indefinitely.

(2) The points $(0, 0)$, $\left(1.57\,a, \frac{\pi}{2}\right)$, $(3.14\,a, \pi)$, $\left(4.71\,a, \frac{3\,\pi}{2}\right)$, $(6.28\,a, 2\,\pi)$, etc., are on the locus [approximately].

(3) If (r_1, θ_1), and $(r_2, \theta_1 + 2\,\pi)$, be points on the locus, then $r_2 = a\theta_1 + 2\,a\pi = r_1 + 2\,\pi a.$, *i.e.*, any radius vector is divided by the curve into equal segments, each of length $2\,\pi a$.

(4) For negative values of θ the locus is symmetrical with the part shown by the full line in the figure, with respect to the line $\theta = \frac{\pi}{2}$.

This curve is the **Spiral of Archimedes.**

(b) *Trace* $$r = a \sin 2\,\theta.$$

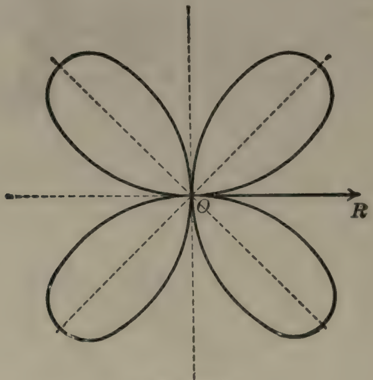

FIG. 123.

(1) The pole is on the curve.

(2) The maximum numerical value of r is obtained if $\sin 2\theta = 1$, *i.e.*, if $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, etc.

(3) r increases from 0 to a as θ increases from 0 to $\frac{\pi}{4}$: then r decreases to 0, as θ increases to $\frac{\pi}{2}$. A loop is formed, symmetrical with respect to the line $\theta = \frac{\pi}{4}$.

(4) The curve consists of four such loops; and is symmetrical with respect to the four lines $\theta = 0$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$, and $\theta = -\frac{\pi}{4}$.

The locus is the **Four-leaved Rose.**

EXERCISES

Trace in detail the following loci:

1. $r = a \sin \theta$.
2. $r(1 - \cos\theta) = 2a$: a parabola.
3. $r = 3\cos\theta + 2$.

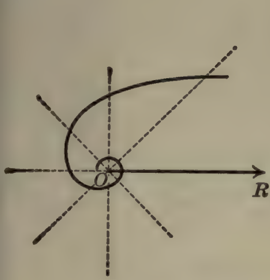

FIG. 124.

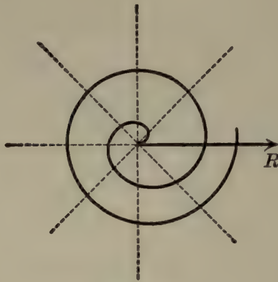

FIG. 125.

4. $r = \dfrac{a}{\theta}$: the **reciprocal spiral.** **5.** $r^2 = a\theta$: the **parabolic spiral.**

FIG. 126.

6. $r^2 = \dfrac{a}{\theta}$: the **lituus.**

7. $r \cos 2\,\theta = a.$

8. $r^2 \cos 2\,\theta = a^2$: an equilateral hyperbola with pole at the center.

9. $r \sin\theta \tan\theta = 4\,a$: a parabola.

10. $r = b - a\cos\theta$: the **limaçon** of Pascal. Show the three forms, for $b < a$, $b = a$, and $b > a$, respectively.

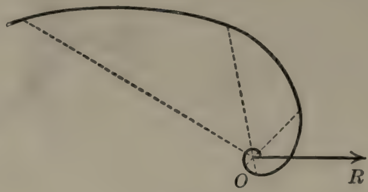

FIG. 127.

11. $r = a^\theta$: the logarithmic spiral.

12. $r = a \sin\theta + b\cos\theta$: a circle.

13. $r = a\sec^2\dfrac{\theta}{2}$.

14. $r = \dfrac{2a\sec\theta}{1+\tan\theta}$.

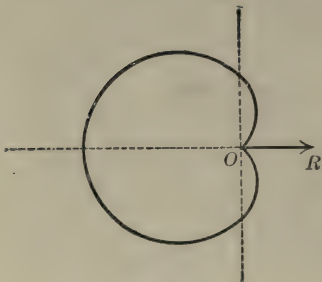

FIG. 128.

15. $r = a(1 - \cos\theta)$: the cardioid.

16 $r^2 = a^2 \sin 2\theta$: the lemniscate.

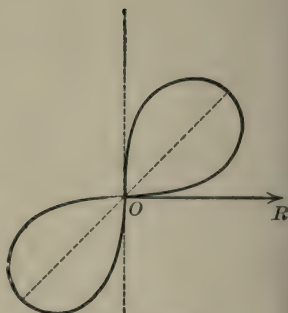

FIG. 129.

17. $r = a \sin^3 \frac{\theta}{3}$.

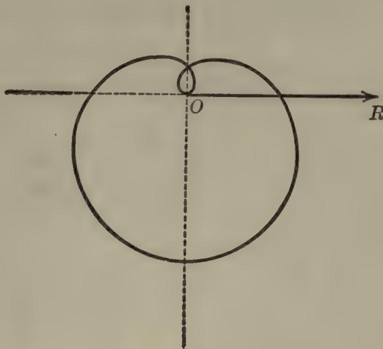

FIG. 130.

169. Parametric equations. Some curves are conveniently represented in Cartesian coördinates, but with a third number — the *parameter* — as the independent variable. Thus, the parametric equations of the ellipse are

$$x = a \cos \phi, \quad y = b \sin \phi,$$

where ϕ is the eccentric angle [Art. 119].

Among the important parametric equations are those of the **cycloid**:

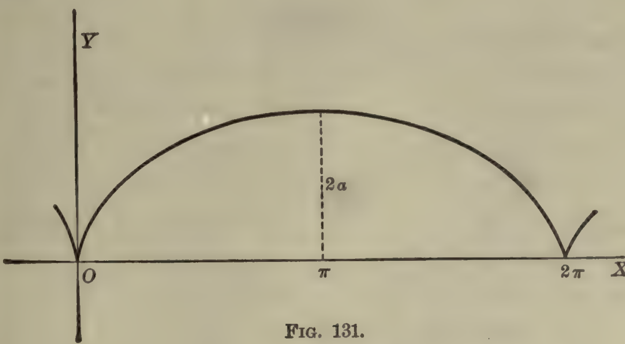

FIG. 131.

$$x = a(\phi - \sin \phi), \quad y = a(1 - \cos \phi).$$

The equations for the simple **hypocycloid** — the **astroid** — are

$$x = a\cos^3\phi,\ y = a\sin^3\phi;$$

and the equations for the **epicycloid**, of two arches,

$$x = \frac{3a}{2}\cos\phi - \frac{a}{2}\cos 3\phi;$$

$$y = \frac{3a}{2}\sin\phi - \frac{a}{2}\sin 3\phi.$$

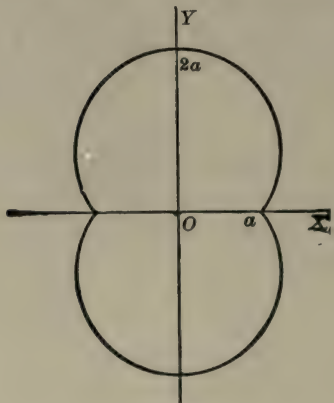

FIG. 132.

EXERCISES

Trace the curves:

1. $x = a\sin\phi,$
   $y = a\cos\phi.$

2. $x = 5 + 2\cos\phi,$
   $y = 2\sin\phi.$

3. $x = b\cos\phi,$
   $y = a\sin\phi.$

4. $x = 5 + 2\cos\phi,$
   $y = 4 + 3\sin\phi.$

5. $x = 2\phi - 2\sin\phi,$
   $y = 2 - 2\cos\phi.$

6. $x = 3\sin\phi - \sin 3\phi,$
   $y = 3\cos\phi - \cos 3\phi.$

7. $x = 6\cos\phi - 4\cos^3\phi,$
   $y = 4\sin^3\phi.$

8. $x = t - a,$
   $y = bt^2.$

9. $x = 2^\phi\sin\phi,\ y = 2^\phi\cos\phi.$

ANSWERS

Pages 6–7. Art. 6.

1. (2), (3), (4), and (5) are identities. **3.** Roots imaginary.

4. $x = -\frac{3+m}{4(2+m)} \pm \frac{1}{4(2+m)}\sqrt{-31m^2-82m-39}$; roots are equal if $m = -\frac{41}{31} \pm \frac{2}{31}\sqrt{118}$.

5. (1) imaginary; (2) imaginary; (3) real and unequal.

6. (1) $m = 2$ or $-\frac{10}{9}$; (2) $m = -\frac{1}{17} \pm \frac{12}{17}\sqrt{-19}$; (3) $m = 3$ or -5; (4) $m = \pm a\sqrt{37}$.

9. (1) $c = \frac{4}{3}$; (2) $c = \frac{l}{m}$; (3) $c = \pm 2\sqrt{10}$.

10. Ex. 2. $x = -\frac{4}{3} \pm \frac{1}{3}\sqrt{13}$; Ex. 3. $x = \frac{3}{16} \pm \frac{1}{16}\sqrt{-23}$;

Ex. 5. $\begin{cases} z = \frac{1}{5} \pm \frac{2}{5}\sqrt{-6}, \\ x = \pm\sqrt{-7}, \\ t = \frac{1}{6} \pm \frac{1}{6}\sqrt{229}. \end{cases}$

11. $x = \pm 4$ or ± 3; $x = -2$ or -5.

Pages 9–10. Art. 9.

1. $x_1 + x_2 = -\frac{3}{1+m}$; $x_1x_2 = -4$;

$$(1+m)\left\{x + \frac{3}{2(1+m)} - \frac{\sqrt{25+32m+16m^2}}{2(1+m)}\right\};$$

$$\left\{x + \frac{3}{2(1+m)} + \frac{\sqrt{25+32m+16m^2}}{2(1+m)}\right\}.$$

2. (1) $(x-4)(x-1)$; (2) $(x+4)(x-2)$;

(3) $m\left\{x-\frac{3}{2m}-\frac{1}{2m}\sqrt{9-4cm}\right\}\left\{x-\frac{3}{2m}+\frac{1}{2m}\sqrt{9-4cm}\right\}$;

(4) $a\left\{x-\left(\frac{b}{2a}+\frac{1}{2a}\sqrt{b^2-4ac}\right)y\right\}\left\{x-\left(\frac{b}{2a}-\frac{1}{2a}\sqrt{b^2-4ac}\right)y\right\}$;

(5) $(3w^{\frac{5}{6}}+2)(w^{\frac{5}{6}}-32)$; (6) $(11+6y)(1-3y)$.

3. $x_1+x_2=-\frac{m+3}{2m+4}$; $x_1x_2=\frac{4m+3}{2m+4}$; $x_1=x_2$ if $m=\frac{-41}{31}\pm\frac{2}{31}\sqrt{118}$; one root becomes infinite if $m \doteq -\frac{1}{2}$;

$$2(m+2)\left\{x+\frac{m+3}{4(m+2)}-\frac{1}{4(m+2)}\sqrt{-31m^2-82m-39}\right\};$$
$$\left\{x+\frac{m+3}{4(m+2)}+\frac{1}{4(m+2)}\sqrt{-31m^2-82m-39}\right\}.$$

4. The roots are equal if $m=\frac{5\pm2\sqrt{-5}}{10}$; the roots are real for all real values of m; one root becomes infinitely great if $m \doteq 2$; one root becomes zero if $m \doteq 0$; the factors are

$$(m-2)\left\{\log x-\frac{2m+3}{2(m-2)}-\frac{1}{2(m-2)}\sqrt{20m^2-20m+9}\right\};$$
$$\left\{\log x-\frac{2m+3}{2(m-2)}+\frac{1}{2(m-2)}\sqrt{20m^2-20m+9}\right\}.$$

6. Real and equal; imaginary.

7. (1) $x=\frac{6}{31}$, $y=\frac{16}{31}$; (2) $y=-\frac{4}{3}$, $z=\frac{11}{6}$;

(3) $x=\frac{3\pm\sqrt{9+12c}}{6}+\frac{c}{3}$, $y=\frac{3\pm\sqrt{9+12c}}{2}$;

(4) $x=-3\pm\sqrt{14}$, $y=\sqrt{-6(3\pm\sqrt{14})}$;

(5) $x=0$ or $-\frac{2a^3b}{b^2+a^4}$, $y=b$ or $\frac{b(b^2-a^4)}{b^2+a^4}$; (6) $x=\pm4$, $y=0$.

8, 9. (1) $b=\pm a\sqrt{37}$; (2) $b=\frac{1}{m}$; (3) $b=\pm2\sqrt{29}$.

Pages 14–15. Art. 12.

1. $15°=\frac{\pi}{12}^{(r)}=0.2618^{(r)}$ approximately; $60°=\frac{\pi}{3}^{(r)}=1.0472^{(r)}$ approximately; etc.

2. $\left(\frac{\pi}{4}\right)^{(r)}=45°$; $\left(\frac{3\pi}{5}\right)^{(r)}=08°$; $\left(\frac{1}{4}\right)^{(r)}=14°+19'+26.2''$; etc.

3. (1) $\sin\theta = \pm\dfrac{3}{\sqrt{10}}$, $\cos\theta = \pm\dfrac{1}{\sqrt{10}}$, $\cot\theta = \frac{1}{3}$, $\sec\theta = \pm\sqrt{10}$, $\csc\theta = \pm\dfrac{\sqrt{10}}{3}$; (2) $\sin x = \pm\dfrac{1}{\sqrt{2}}$, $\cos x = -\dfrac{1}{\sqrt{2}}$, $\tan x = \mp 1$, $\cot x = \mp 1$, $\csc x = \pm\sqrt{2}$; etc.

4. $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, etc.; $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$, etc.; $\sin 60° = \cos 30° = \frac{1}{2}\sqrt{3}$, $\cos 60° = \frac{1}{2}$, etc.; $\sin 90° = 1$, $\cos 90° = 0$, etc.; $\sin 135° = \dfrac{1}{\sqrt{2}}$, $\cos 135° = -\dfrac{1}{\sqrt{2}}$, etc.; $\sin(-45°) = -\dfrac{1}{\sqrt{2}}$, $\cos(-45°) = \dfrac{1}{\sqrt{2}}$, etc.

5. $\tan 3500° = \tan 80° = \cot 10°$; $-\csc 290° = \csc 70° = \sec 20°$; $\sin(-369°) = -\sin 9°$; $-\cos\dfrac{11\pi}{5} = -\cos\dfrac{\pi}{5}$; $\cot(-1215°) = \cot 45°$.

6. (1) $\sin\theta = -\cos 210° = \cos 30° = \sin 60°$, hence one value of θ is $60°$; (2) $\theta = 30°$; (3) $x = 60°$; (4) $x = \pm 30°$.

Pages 17–18. Art. 13.

1. $\frac{3}{2}$, $\frac{7}{2}$. **2.** $\frac{5}{2}$, 0, $\frac{5}{2}$. **3.** $\dfrac{8}{\sqrt{13}}$, $\dfrac{18}{\sqrt{13}}$.

4. $\dfrac{a}{\sqrt{3}}$ if a is the edge of the cube; $\frac{15}{2}$, $15\sin\alpha$. **5.** $b\sin\omega$. **6.** 0.

Pages 22–23. Art. 17.

3. Point of intersection is (0, 0); middle point is (0, 0).

4. On the axis of abscissas (x-axis); on the axis of ordinates (y-axis); on the line bisecting the 1st and 3d angles formed by the coördinate axes; on the line bisecting the 2d and 4th angles formed by the coördinate axes.

5. $y = 0$, $x = 0$, $x - y = 0$, $x + y = 0$.

6. $(-\frac{5}{2}, 0)$, $(\frac{5}{2}, 0)$, and $(0, \frac{5}{2}\sqrt{3})$; $(0, 0)$, $(5, 0)$, and $(0, 5)$.

7. $(5\sqrt{2}, 0)$, $(0, 5\sqrt{2})$, $(-5\sqrt{2}, 0)$, and $(0, -5\sqrt{2})$; $(0, 0)$, $(0, 10)$, $(10\sqrt{2}, 0)$, and $(10\sqrt{2}, -10)$.

8. $(\frac{3}{2}, -1)$. **9.** $\sqrt{a^2+b^2}$; $\sqrt{a^2+b^2}$; $\sqrt{a^2+b^2}$.

Pages 25–27. Art. 21.

1. $2\sqrt{17}$, $2\sqrt{157}$, $2\sqrt{74}$.

2. $2\sqrt{17+4\sqrt{3}}$, $2\sqrt{157+66\sqrt{3}}$, $2\sqrt{74+35\sqrt{3}}$, with $\omega = 30°$;
$2\sqrt{17+4\sqrt{2}}$, $2\sqrt{157+66\sqrt{2}}$, $2\sqrt{74+35\sqrt{2}}$, with $\omega = 45°$.

4. $\sqrt{(b-c)^2+(a-b)^2}$, $2\sqrt{a^2+b^2}$.

6. Yes.

7. 16 or -8.

8. $\sqrt{(x+4)^2+(y-6)^2}=5$, *i.e.* $x^2+y^2+8x-12y+27=0$; $x^2+y^2=25$.

9. $8x+2y=31$. 10. 1; $\frac{12}{5}$; $-\frac{9}{4}$; $-\frac{1}{6}$.

12. $2x-3y+10=0$.

Page 29. Art. 23.

1. (1) 4; (2) $33\frac{1}{2}$. 3. Yes.

Pages 31–32. Art. 24.

3. $(\frac{8}{3}, \frac{1}{3})$ and $(\frac{10}{3}, -\frac{7}{3})$. 6. $(\frac{14}{5}, \frac{58}{5})$ and $(-2, 2)$. 7. $C\equiv(8, 0)$.

Pages 32–34. Examples on Chapter II.

1. $13\frac{1}{2}$. 2. Base is $5\sqrt{2}$, sides are $\sqrt{97}$, alt. is $\frac{13}{2}\sqrt{2}$, area is $32\frac{1}{2}$.

3. $(\frac{5}{7}, -\frac{6}{7})$; $(8, 21)$; $(-7, -24)$. 4. $(7, 9)$. 5. $\frac{1}{9}\sqrt{5218}$.

8. Slopes are: 1, $\frac{1+a}{1-a}$, $\frac{1-a}{1+a}$.

11. $\sqrt{(h+1)^2+(k-1)^2}=\sqrt{(h-1)^2+(k-2)^2}$, *i.e.* $4h+2k=3$; $k=0$. 13. $(1, -11)$, $(-11, 5)$ or $(13, -1)$. 14. $(3, -\frac{8}{3})$.

15. $\left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, $\left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$.

18. $3\sqrt{13}+6\sqrt{3}+3$. 20. $(\pm\frac{7}{5}\sqrt{5}, \mp\frac{14}{5}\sqrt{5})$. 22. $\frac{bc}{a}$.

Page 45. Art. 32.

1. $12y-13x+71=0$; $3y-4x+23=0$.

2. $4y-6x=0$; $(0, 0)$. 3. $y-x+24=0$; $y+x+4=0$.

4. $\sqrt{3}y+x-2\sqrt{3}-6=0$. 5. $x^2+y^2-10y=0$.

Page 47. Art. 34.

1. $x^2-8x+y^2-9=0$. 2. $18x+4y=61$.

3. $x^2-4x+y^2+6y+4=0$. 4. $y^2=16(x-4)$.

5. $16x^2+11y^2=176$. 6. $4y^2-x^2=4$.

Pages 49–50. Art. 36.

1. $(\frac{44}{9}, \frac{28}{3})$. 2. $(4, 0)$. 3. $(0, 2), (-\frac{6}{5}, -\frac{8}{5})$.

4. $(2\sqrt{-\frac{2}{7}}, 5\sqrt{-\frac{2}{7}}), (-2\sqrt{-\frac{2}{7}}, -5\sqrt{-\frac{2}{7}})$.

5. Two of the four points are : $\left(\frac{\pm\sqrt{6}(1+\sqrt{2})}{2}, \frac{\pm\sqrt{6}(1-\sqrt{2})}{2}\right)$.

6. Two of the four points are: $\left(\pm\frac{\sqrt{34}+4}{2}, \pm\frac{\sqrt{34}-4}{2}\right)$.

7. $(0, 0), (4, 4)$.

8. $\left(\frac{ab}{a^2+b^2}[2a \pm \sqrt{a^2-3b^2}], \frac{b}{a^2+b^2}[2b^2 \mp a\sqrt{a^2-3b^2}]\right)$.

9. $(\pm\sqrt{11}, \pm\sqrt{5})$. 10. $(-4, 4), (8, 16)$.

Page 51. Art. 37.

1. The two axes, *i.e.* the loci of $y = 0$ and $x = 0$.

2. The loci of $\frac{x}{3} + \frac{y}{2} = 0$ and $\frac{x}{3} - \frac{y}{2} = 0$.

3. $x = 0$ and $3x + 2y = 7$. 4. $x = 0$, $y = 0$, and $2y - 5x = 0$.

5. $x + 2 = 0$ and $x + 2 = 0$.

6. $x^2 + y^2 = 10$, $y + \sqrt{10}x = 0$, $y - \sqrt{10}x = 0$.

Page 54. Art. 39.

2. (α) $x^2 + y^2 - 4 + k(x^2 + 2x + y^2) = 0$; ($\beta$) $y - 2\sin x + k(y - \cos x) = 0$; ($\gamma$) $x^2 - 4y + k(y^2 - 4x) = 0$.

Pages 54–58. Examples on Chapter III.

1. The first and third are; the second is not. 2 Yes.

3. ± 4; $\pm\sqrt{25 - a^2}$.

4. This curve cuts the x-axis in the points $(-2, 0)$ and $(-3, 0)$; it cuts the y-axis in the point $(0, 6)$.

5. The x-intercepts are ± 3, and the y-intercepts are $\pm 4\sqrt{-1}$.

6. The x-intercepts are $-3 \pm 2\sqrt{3}$, and the y-intercepts are $2 \pm \sqrt{7}$.

7. The four points are : $(\pm\sqrt{10}, \pm\frac{3}{2}\sqrt{6})$; the lengths of the sides are: $2\sqrt{10}$ and $3\sqrt{6}$; and the lengths of the diagonals are $\sqrt{94}$.

8. $12a^2$. 9. $\frac{18}{13}\sqrt{13}$. 10. Yes. 11. Yes.

12. For $m^2 > 333$.

13. $\begin{cases} \text{Distinct points if } b < 1. \\ \text{Coincident points if } b = 1 \\ \text{Imaginary points if } b > 1. \end{cases}$

14. $c = \pm 6\sqrt{10}$.

15. $x^2 + y^2 - 25 + k(y^2 - 4x) = 0$.

16. $(3x - 2y + 12)(y^2 - 4x)(x^2 + y^2 - a^2) = 0$.

24. $(3, 0)$ and $(-3, 0)$. **25.** $(0, 1)$. **26.** $4x + y = 11$, $x - y = 9$, and $9x + y = 21$. **27.** $y - x = 1$, $x + y = 3$, $x - y = 3$, $x + y + 1 = 0$.

28. Center is at $(1, 0)$, radius is 2, eq. of circle is $x^2 - 2x + y^2 = 3$.

29. (α) $x + y = 0$; (β) $3x + y = 17$; (γ) $x - y = 0$.

30. $2x - 3y = 12$. **31.** $x^2 = 4y$. **32.** $9x^2 + 25y^2 - 54x - 150y + 81 = 0$.

33. $15x^2 - y^2 - 90x + 6y + 111 = 0$. **34.** $x^2 = 9y^2$.

35. $(x - a)^2 + y^2 = 4y$. **36.** $3x^2 - y^2 = 0$. **37.** $6x - 2y = 15$.

38. $2x^2 + 2y^2 - 10x - 10y + 25 = 0$. **39.** $6ax + 2bx = b^2 - a^2$.

40. $5y - 4x = 20$. **41.** $4x + 14y = 9$; $2x - 8y = 21$; $(\frac{61}{16}, -\frac{11}{16})$.

42. $6x - 3y + 14 = 0$. **43.** $5x + 4y + 24 = 0$, or $5x + 4y - 26 = 0$.

44. $5y - 4x = 31$.

Pages 61–62. Art. 42.

2. (α) $y + 2x = 10$; (β) $y + 6x = 22$; (γ) $3x + 2y + 16 = 0$; (δ) $159y - 20x + 177 = 0$.

3. (α) $\frac{x}{7} + \frac{y}{4} = 1$; ($\beta$) $\frac{y}{10} - \frac{x}{6} = 1$; ($\gamma$) $\frac{3y}{4} - 2x = 1$;

(δ) $\frac{x}{3a} + \frac{2y}{a} + 1 = 0$.

6. Yes. **7.** Yes **8.** No; the point and the origin are on the same side of the line.

9. $-\frac{13}{4}$, $-\frac{18}{11}$.

10. Equations of the sides are: $\begin{cases} 4x + y = 22, \\ 9x + y = 42, \\ x - y = 18. \end{cases}$

Equations of the medians are: $\begin{cases} x = 6, \\ 7x + 3y = 26, \\ 17x + 3y = 86. \end{cases}$

Medians intersect in the point $\left(6, \frac{-16}{3}\right)$.

11. $-\frac{4}{11}$. **12.** $x - 3y = 0$, $4x - 3y = 0$.

Pages 63–64. Art. 43.

4. $y = \dfrac{x}{\sqrt{3}} + 7 + \sqrt{3}$; $y = -\sqrt{3}\,x + 7 - 3\sqrt{3}$; $y = -\sqrt{3}\,x + 7 - 3\sqrt{3}$;
$y = -\dfrac{x}{\sqrt{3}} + 7 - \sqrt{3}$.

5. (α) $y = 4x + 4$; (β) $y = \dfrac{x}{2} - 3$; (γ) $y = -2x - \dfrac{3}{2}$.

6. 43. 7. $-\frac{2}{5}$. 8. Yes.

9. They differ in their y-intercepts, but have the same slope.

10. The y-intercept. 11. $\dfrac{y_2 - y_1}{x_2 - x_1}$; $-\dfrac{b}{a}$.

Pages 65–66. Art. 44.

1. $\sqrt{3}\,x + y = 12$. 3. $x\sqrt{3} + y = 8$. 4. $\frac{9}{2}\sqrt{2}$. 5. $x + y + 7 = 0$.

6. $\alpha = 135°$; $p = 0$. 7. Yes.

Pages 70–72. Art. 47.

1 (α) $\dfrac{x}{-6} + \dfrac{y}{4} = 1$; ($\beta$) $\dfrac{x}{-\frac{1}{2}} + \dfrac{y}{-\frac{1}{2}} = 1$; ($\gamma$) $\dfrac{x}{-3} + \dfrac{y}{5} = 1$;
(δ) $\dfrac{x}{26} + \dfrac{y}{-\frac{2}{5}} = 1$.

2. (α) $y = x + 4$; (β) $y = -\frac{3}{2}x - 3$; (γ) $y = -\frac{1}{3}x + \frac{8}{3}$.

3. (α) $x\frac{4}{5} + y\frac{3}{5} = 3$; ($\beta$) $x(-\frac{4}{5}) + y\frac{3}{5} = 3$;
(γ) $\dfrac{5x}{\sqrt{34}} + \dfrac{3y}{\sqrt{34}} = \dfrac{-5}{\sqrt{34}}$; ($\delta$) $-\dfrac{2x}{\sqrt{13}} + \dfrac{3y}{\sqrt{13}} = \dfrac{15}{\sqrt{13}}$.

5. Slope is -3; distance from origin is $\dfrac{8}{\sqrt{10}}$.

6. $x + y = 11$; $p = \dfrac{11}{\sqrt{2}}$. 7. $\frac{1}{2}$. 8. $a = \dfrac{25}{7}$, $b = -\dfrac{25}{4}$; $a = 0$, $b = 0$.

9. A system of parallel lines of slope 8.

10. All the lines passing through the point (0, 10).

11. All the tangents to the circle of radius 10, and center at the origin.

12. (α) $\cos\alpha = \dfrac{-m}{\sqrt{1+m^2}}$, $\sin\alpha = \dfrac{1}{\sqrt{1+m^2}}$;
(β) $\cos\alpha = \dfrac{-b}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$;
(γ) $\cos\alpha = \dfrac{-3}{\sqrt{13}}$, $\sin\alpha = \dfrac{2}{\sqrt{13}}$;
(δ) $\cos\alpha = \dfrac{-7}{\sqrt{74}}$, $\sin\alpha = \dfrac{5}{\sqrt{74}}$.

13. (α) second; (β) third; (γ) first. **14.** $-\frac{5}{4}$; no.

15. $A : B : C = 1 : -4 : 12$. **17.** $\frac{3}{5}$ and $-\frac{5}{3}$. **18.** $\dfrac{9}{\sqrt{82}}$ and $\dfrac{-1}{\sqrt{82}}$.

Page 73. Art. 48.

1. $-\dfrac{1}{2}$. **2.** $-\dfrac{22}{19}$. **3.** 0. **4.** $\dfrac{-2ab}{a^2-b^2}$. **5.** $\dfrac{-b\sin\alpha + a\cos\alpha}{b\cos\alpha + a\sin\alpha}$.

Pages 75–76. Art. 49.

1. (α) $y = 6x + k$; (β) $3x - 7y = k$; (γ) $x\cos 30° + y\sin 30° = k$; ($\delta$) $\dfrac{x}{a} - \dfrac{y}{b} = k$, — where k may have any value whatever.

3. $\dfrac{39}{25}$; $\dfrac{3a-7b}{7a+3b}$; $\dfrac{6a-b}{a+6b}$; from the second line to the first in each case.

4. (α) $x + 6y = k$; (β) $7x + 3y = k$; (γ) $x\sin 30° - y\cos 30° = k$; ($\delta$) $ax + by = k$.

5. $6x - y + 55 = 0$. **7.** $Ax + By = Ax_1 + By_1$. **8.** $2y = x - 12$.

9. $x + y + 2 = 0$. **10.** $125x - 116y = 9$.

11. $196x - 56y = 45$. **12.** $\left(\dfrac{35}{37}, \dfrac{-25}{37}\right)$.

Pages 78–81. Art. 50.

1. $5x + y = 45$, $5y - x = 17$. **3.** $7x + y = b$. **4.** $y = \dfrac{12\sqrt{3}+5}{5\sqrt{3}-12}x + b$.

5. $y - 3 = \dfrac{2+\sqrt{3}}{2\sqrt{3}-1}(x-1)$, $y - 3 = \dfrac{\sqrt{3}-2}{2\sqrt{3}+1}(x-1)$. **6.** $x=0$; $y=0$.

7. $y + \dfrac{b}{2} = \dfrac{2ab}{a^2-b^2}\left(x - \dfrac{a}{2}\right)$. **8.** $x + 2y = 14$ or $x - 2y + 6 = 0$.

11. $x = 1$, $y = 1$, $x - y = 0$. **12.** $16x - 5y + 47 = 0$.

14. (1) $3x - 2y + 3 = 0$; (2) $2x + 3y = 11$;
(3) $(2\sqrt{3} - 3)x + (3\sqrt{3} + 2)y = 11\sqrt{3} + 3$; (4) $y = 3$.

16. Yes; $\tan^{-1}\frac{1}{17}$, $\tan^{-1}\frac{1}{12}$, $\tan^{-1}\frac{1}{7}$.

17. 90°, 45°, 45°. **18.** Parallel if $b = 0$; perpendicular if $b^2 = a^2 + 1$.

19. Distance between lines is $\dfrac{24}{\sqrt{17}}$. **20.** $\dfrac{5}{2\sqrt{34}}$. **21.** $\dfrac{25}{\sqrt{629}}$.

22. They make numerically equal but opposite angles with the x-axis

23. $\tan^{-1}(\pm\frac{33}{56})$; $7y = 9x - 1$, $7x + 9y = 73$.

24. 45°; $(8, -6)$; $y = 3x + 9$, $2y = x + 8$.

Pages 83–84. Art. 51.

2. $\frac{20}{\sqrt{41}}$. 3. $\frac{17}{2\sqrt{13}}$. 4. $\frac{-3ab}{\sqrt{a^2+b^2}}$. 5. $\frac{-6}{\sqrt{85}}$. 6. $-\frac{24}{\sqrt{5}}$

7. $-\frac{47}{\sqrt{58}}$. 8. $\frac{7}{\sqrt{29}}$; $\frac{C}{A^2+B^2}$. 9. $\frac{45}{\sqrt{58}}$. 10. $\frac{y_1-mx_1-b}{\sqrt{1+m^2}}$.

11. Altitudes are: $\frac{36}{\sqrt{10}}, \frac{36}{\sqrt{34}}, \frac{18}{\sqrt{13}}$; the area is 36.

13. $\frac{x_1-2y_1-11}{\sqrt{5}} = \pm\frac{3x_1-4y_1-5}{5}$.

14. $y-x=12$, and $7x+7y=36$.

Page 85. Art. 52.

1. $\frac{x-y+6}{\sqrt{2}} = \pm\frac{3x-10y+10}{\sqrt{109}}$.

2. Other bisector is $7x-7y+24=0$.

5. $\phi_1 = \tan^{-1}\left(\frac{3b-a}{b+3a}\right)$; $\phi_2 = \tan^{-1}\left(\frac{2a-b}{a+2b}\right)$;

$$a = \frac{38\pm 15\sqrt{2}}{142},\ b = \frac{16\pm 25\sqrt{2}}{142}.$$

Page 87. Art. 53.

1. $3y+x+6=0$, $2y-x+6=0$; $(\frac{6}{5}, -\frac{12}{5})$; 45°.
2. $x-3y+1=0$, $x+y+1=0$; $(-1, 0)$; $\tan^{-1}2$.
3. $x-y(\sec\alpha+\tan\alpha)=0$, $x-y(\sec\alpha-\tan\alpha)=0$; $(0, 0)$; α.
4. $x+3y+5=0$, $x+3y-1=0$; lines are parallel.
5. $6x-5y+14=0$, $6x+5y=56$; $(\frac{7}{2}, 7)$.

Pages 88–91. Examples on Chapter IV.

1. The third vertex is $(2\pm 4\sqrt{3}, \pm 4\sqrt{3})$; equations of the two sides are: $(1\pm\sqrt{3})y + (1\mp\sqrt{3})x - 2\mp 6\sqrt{3} = 0$, and

$$(1\pm\sqrt{3})x + (1\mp\sqrt{3})y - 1\mp 5\sqrt{3} = 0.$$

2. The area is 72, and the fourth vertex is at any one of the following points: $(6, -10)$, $(14, 2)$, $(-2, 14)$.

3. $(9\pm 3\sqrt{3})x - 12y - 3(9\pm 3\sqrt{3}) = 0$. 5. $x=1$.

6. $5x-2y=50$. 7. $y=(2\pm\sqrt{3})x\mp\sqrt{3}$. 8. $y=0$ or $x=5$.

11. If the base be chosen as x-axis, and its middle point as origin, then the equation of the locus of the vertex is $ax - 2c^2 = 0$.

12. $2x - 5y + 7 = 0$, $x - y = 1$; $1:1:1$.

13. $(7\sqrt{10} - 3\sqrt{53})y + (\sqrt{53} - 2\sqrt{10})x = 11\sqrt{53} + \sqrt{10} \pm 4\sqrt{530}$.

14. $4\sqrt{53}(3y - x + 11) = 3\sqrt{10}(2x - 7y + 1)$.

15. $y = \sqrt{3}x + 3 - \sqrt{3}$. **16.** $x + y = 0$. **17.** $3x + 2y = 13$.

18. Center is $(7, 2)$, radius is 5.

19. If the base $(= 2a)$ coincides with the x-axis, its middle point at the origin, and if $k \equiv$ the area of the triangle, then the locus of the vertex is $ay - k = 0$.

21. $4x = 5$ (axes chosen as in Ex. 19).

22. (α) $x = 0$, $y = 0$, $x - y = 0$; (β) $7x + y = 0$, $2x - y = 0$; (γ) $x + a = 0$, $y + b = 0$. **23.** $m = 3$.

26. $9y - 4x = 24$, $4x - y = 8$. **27.** $20y - 9x = 96$. **28.** $x = 6$.

29. $x^2 - y^2 = 0$. **30.** $x = 4$, $x = 6$, $y = 6$, $y = 8$.

32. $\tan^{-1}\left(\dfrac{2\sqrt{R^2 - PQ}}{P + Q}\right)$.

Page 93. Art. 55.

1. $x^2 + y^2 - 12x + 8y + 27 = 0$. **2.** $4x^2 + 4y^2 + 24y + 27 = 0$.

3. $x^2 + y^2 - 8x + 8y + 16 = 0$. **4.** $x^2 + y^2 = 25$.

5. $x^2 + y^2 + 8y + 7 = 0$. **7.** $\begin{cases} x^2 + y^2 \pm 2hx = 0; \\ x^2 + y^2 + 2h(x - y) + h^2 = 0 \end{cases}$

Pages 95–96. Art. 57.

1. $r = \frac{3}{2}\sqrt{21}$; $(\frac{3}{2}, 2)$. **2.** $r = \frac{1}{6}\sqrt{62}$; $(\frac{5}{6}, \frac{7}{6})$. **3.** $r = \frac{3}{2}\sqrt{5}$; $(\frac{3}{2}, 0)$

4. $r = \frac{7}{4}$; $(0, \frac{7}{4})$. **5.** $r = \dfrac{1}{2a}\sqrt{b^2 + c^2}$; $\left(-\dfrac{b}{2a}, -\dfrac{c}{2a}\right)$.

6. $r = 2a$; $(0, 0)$. **7.** A point circle; imaginary circle.

8. $x^2 + y^2 - 22x - 4y + 25 = 0$.

9. $(a + b)(x^2 + y^2) - (a^2 + b^2)(x + y) = 0$.

10. $x^2 + y^2 - 14x - 4y - 5 = 0$. **11.** $x^2 + y^2 + 7x + 6 = 0$.

12. $x^2 + y^2 - 6x - y + 3 = 0$.

13. $x^2 + y^2 - 40(x + y) + 400 = 0$, $x^2 + y^2 - 8(x + y) + 16 = 0$.

14. $9(x^2 + y^2) - 42y + 47 = 0$, center is $\left(0, \dfrac{7}{3}\right)$, $r = \dfrac{\sqrt{2}}{3}$; or

$8(x^2 + y^2) + 8x - 48y + 73 = 0$, center is $\left(-\dfrac{1}{2}, 3\right)$, $r = \dfrac{\sqrt{2}}{4}$.

Page 99. Art. 59.

1. $x + 2y \pm 2\sqrt{5} = 0$.
2. $3y = 3x + 2 \pm 2\sqrt{2}$.
3. $y = 2x + 13 \pm 6\sqrt{5}$.
4. $x \pm y = \pm r\sqrt{2}$.
5. $\left(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$.
6. (1, 0) and (0, 1).
7. $12 \pm 20\sqrt{6}$.
8. $\frac{1}{2}\sqrt{10}$.
9. $(0, a)$.
10. $y = -5$, $y = \sqrt{3}x + 10$, $y = -\sqrt{3}x + 10$, area, $75\sqrt{3}$, for one of the four triangles.

Pages 100–101. Art. 60.

1. $y = 4x \pm 5\sqrt{17}$.
2. $3y + x \pm 2\sqrt{10} = 0$.
3. $3y = \sqrt{3}x \pm 6\sqrt{3}$; $y + \sqrt{3}x = \pm 6$.
5. $y = 5x \pm 5\sqrt{26}$.
6. $y = x \pm \sqrt{2}$.
7. $y + F = m(x + G) \pm \sqrt{G^2 + F^2 - c} \cdot \sqrt{1 + m^2}$.

Pages 104–105. Art. 62.

2. (α) $4x + 5y = 41$; (β) $5x - 12y = 152$; (γ) $3x + y + 19 = 0$; (δ) $y + 2x = 0$.
3. (α) $5x = 4y$; (β) $12x + 5y + 7 = 0$; (γ) $x - 3y = 7$; (δ) $x - 2y = 0$.
5. $x - 4 = 0$.
6. $x + 3y = 20$; $31x - 27y = 260$.
7. $13x^2 + 13y^2 - 130x - 78y - 519 = 0$.
8. $\frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{r^2}$.
9. $$\left(x - \frac{a + b - \sqrt{a^2 + b^2}}{2}\right)^2 + \left(y - \frac{a + b - \sqrt{a^2 + b^2}}{2}\right)^2 = \left(\frac{a + b - \sqrt{a^2 + b^2}}{2}\right)^2.$$
10. $(x_1, y_1) \equiv (2, 6)$ or $\equiv \left(\frac{62}{13}, -\frac{54}{13}\right)$.

Page 107. Art. 64.

1. $28\frac{3}{5}$; $26\frac{2}{5}$; $11\frac{11}{12}$; $-4\frac{7}{12}$.
2. $3\frac{3}{4}$; $-2\frac{1}{4}$; 5; +4.
3. $6\frac{2}{3}$; $-5\frac{1}{3}$; 5; +3.
4. $2x^2 + 2y^2 = 51$; $\left(\pm 3, \pm\sqrt{\frac{33}{2}}\right)$.

Page 109. Art. 67.

1. $x - 4y - 3 = 0$.
2. $(-1, -1)$, $(7, +1)$; $\sqrt{68}$.
3. $x - y = 0$; $\frac{1}{2}\sqrt{2(a + b)^2 - 16c}$.
4. $(-2, -1)$.
7. 90°.
8. $\tan^{-1}\sqrt{19}$.

Pages 110–114. Examples on Chapter V.

1. $x^2 + y^2 - 14x - 4y - 5 = 0$; $(7, 2)$; $\sqrt{58}$.

2. (a, b); this family consists of all circles of radius $\sqrt{a^2 + b^2}$, and having their centers on the circle whose equation is $x^2 + y^2 = a^2 + b^2$.

3. $G_1 = G_2$, and $F_1 = F_2$; $G_1^2 + F_1^2 - C_1 = G_2^2 + F_2^2 - C_2$.

4. $D^2 = E^2 = 4F$. 5. $2x^2 + 2y^2 + 13x = 0$. 6. $(2 \pm 3\sqrt{-5}, -8)$.

7. The circles are: $x^2 + y^2 - 14x - 4y = -\frac{26647}{1024}$, $x^2 + y^2 + 2x + 8y = \frac{10217}{1024}$, $x^2 + y^2 - 6x - 6y = \frac{9193}{1024}$.
The radical axes are: $8x - 2y = 35$, $4x + 3y = 9$, $8x + 14y = 1$; radical center $(\frac{123}{32}, -\frac{17}{8})$.

8. $x^2 + y^2 - 2x + y = 15$, $x^2 + y^2 - 6x + 2y = 15$, $x^2 + y^2 - 10x - 5y + 27 = 0$; $4x - y = 0$; $4x + 7y = 42$, $4x + 3y = 21$; $(\frac{21}{16}, \frac{21}{4})$.

9. $y_1(x^2 + y^2) = y(x_1^2 + y_1^2)$. 10. $x + y + 2 = 0$.

11. $27x^2 + 27y^2 - 384x - 134y + 264 = 0$. 12. $5x^2 + 5y^2 = 9$.

13. $ax - by = b^2$, $by - ax = a^2$, $ax + by = 0$.

14. $(x - 16)^2 + (y - 1)^2 = 74$, $(x - 6)^2 + (y - 15)^2 = 74$.

15. $x^2 + y^2 + 4ax - 4ay + 4a^2 = 0$. 16. $x^2 + y^2 = x_1 x + y_1 y$.

20. $(\frac{1}{2}, \frac{5}{2})$. 21. $G_1 : F_1 :: G_2 : F_2$. 22. $2x^2 + 2y^2 + 40x - 85y - 3 = 0$.

23. $y = 2$, $x = 1$, $4x - 3y = 10$, $3x + 4y = 5$.

24. The radical axis is $2x + y = 2$, the ratio is $3 : 7$.

25. $(\frac{33}{4}, 2)$, $\frac{1}{4}$; this point is the radical center.

26. $(r^2 - 1)(x^2 + y^2) - 2(G + ar^2)x - 2(F + br^2)y - C + r^2(a^2 + b^2) = 0$, where r is the given ratio and (a, b) the fixed point.

27. $\sqrt{4r^2 - 2(a - b)^2}$; $r = \dfrac{a - b}{\sqrt{2}}$.

28. $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$; $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$; 3, 0; and for three other points in each case.

29. Equations of sides of inscribed triangle are: $2x + 5 = 0$, $x \pm \sqrt{3}y = 5$; equations of sides of circumscribed triangle are: $x = 5$, $x = \pm\sqrt{3}y - 10$; and the length of the sides are, respectively, $5\sqrt{3}$ and $10\sqrt{3}$.

30. $x^2 + y^2 - 2ax - 2(a + 1)y + 6a - 1 = 0$.

31. $x^2 + y^2 = 2(c^2 - 6a^2)$. 32. $x^2 + y^2 = ax$.

33. $\left(x - \dfrac{x_1}{2}\right)^2 + \left(y - \dfrac{y_1}{2}\right)^2 = \dfrac{a^2}{4}$.

34. $5x^2 + 5y^2 - 50x - 46y + 22 = 0$. 35. Yes.

36. $x^2 + y^2 + 10x + 10y = 35$, or $x^2 + y^2 - 20x - 8y + 31 = 0$.

37. $y - 3 = \dfrac{49 \pm \sqrt{97}}{48}(x - 2)$. 38. $3x + 2y = 13$, $2x - 3y = 0$; $x = \pm\sqrt{\frac{23}{13}}\,y + 6$, $y = 0$. 39. $45°$.

40. Equation of diameter is $5x + 4y = 1$, equation of chords is $5y = 4x + k$.

41. $x^2 + y^2 = r_1^2 + r_2^2$.

Page 117. Art. 69.

3. $(5, 0°)$, $(5, 60°)$. 4. $\left(-2, \dfrac{5\pi}{4}\right)$, $\left(-2, \dfrac{-3\pi}{4}\right)$, $\left(2, \dfrac{-7\pi}{4}\right)$; $(3, 255°)$, $(3, -105°)$, $(-3, -285°)$; $(5, 360°)$, $(5, -360°)$, $(5, -0°)$; $(0, 240°)$, $(0, -120°)$, $(0, -300°)$.

5. On circumference of a circle of radius 7 and center at the pole; same circle; on the line through the pole making an angle of 25° with the initial line; on the initial line; on the initial line.

6. $\rho = 7$; $\rho = -7$; $\theta = 25°$; $\theta = 0^{(r)}$; $\theta = -180°$.

7. Parallel to the initial line.

Pages 119–120. Art. 70.

1. $\sqrt{13 - 12\cos\dfrac{7\pi}{12}}$, $\sqrt{10}$, $\sqrt{5 - 4\cos\dfrac{13\pi}{12}}$. 4. $\dfrac{r^2}{4}$.

5. $14 - \frac{15}{4}\sqrt{3}$; $\sqrt{19} + \sqrt{89} + \sqrt{73 - 24\sqrt{3}}$. 6. $(0, 0)$. 7. $(\pm a, 0)$.

8. $\rho = \tan\theta$. 9. $(6, \cos^{-1}\frac{2}{3})$. 10. ρ and θ of the points of intersection satisfy the equations: $\rho^2\sqrt{2} = 9(1 + \sqrt{\rho^2 - 1})$ and $\theta = \sin^{-1}\left(\dfrac{1}{\rho}\right)$.

14. $\rho - 2\cos\theta + k(\rho\cos\theta - \frac{1}{4}) = 0$.

Pages 122–123. Examples on Chapter VI.

2. $\rho\sin\theta = \pm 3$; $\rho\cos\theta = \pm 3$. 3. $\rho\cos\theta = 5$.

4. $\rho\sin(\phi - \theta) = \rho_1\sin(\phi - \theta_1)$. 5. $(\rho_1, \theta_1) \equiv \left(10, \dfrac{\pi}{2}\right)$.

9. $\rho^2 - 8\rho\cos\left(\theta - \dfrac{\pi}{3}\right) = 9$; $(2 \pm \sqrt{13}, 0)$.

10. $\rho^2 - 20\rho\sin\theta + 96 = 0$; $\sin^2\theta = \frac{24}{25}$.

11. $\rho^2 - 16\rho\sin\theta + 48 = 0$; 8. 12. (α) $\left(\dfrac{a}{\cos 20°}, \dfrac{2\pi}{9}\right)$; (β) $\left(\dfrac{a}{2}\sqrt{3}, \dfrac{\pi}{3}\right)$.

Page 126. Art. 73.

1. $2x - 3y = 0$. 2. $x^2 + y^2 = 30$. 3. $y^2 - 4x^2 = 4$. 4. $y = mx$.

Page 127. Art. 74.

1. $x^2+y^2=a^2$. 2. $2xy+25=0$. 3. $x=5\sqrt{2}$
4. $9x^2+25y^2=225$. 5. $x^2-4y^2=a^2$.

Pages 129–130. Art. 76.

1. $2\rho=a$. 2. $\rho\sin\theta-\cot\theta+4a=0$. 3. $25\cos 2\theta=1$.
4. $m\tan\theta=n$. 5. $\rho(\cos\theta+\sqrt{3}\sin\theta)=2$. 6. $\rho^2\cos 2\theta=49$.
7. $x^2+y^2=a^2$. 8. $x^2-y^2=4$. 9. $xy=10$.
10. $(x^2+y^2)^2=2a^2xy$. 11. $x^2+y^2=10x$. 12. $11y=2x$.
13. $y^2+4kx=4k^2$.

Pages 130–133. Examples on Chapter VII.

1. $xy=4$. 2. $\sqrt{5}y^2+6y=0$. 3. $x^2-xy=0$.
4. $\sqrt{2}(y^2-xy-2)+x+3y=0$. 6. $(-3,-1)$. 7. $x+y=0$, $3x-2y=0$.
8. $y=0$. 9. $\tan^{-1}(\frac{1}{3})$. 11. $\tan\theta=-\frac{A}{B}$; $\tan\theta=\frac{B}{A}$.
13. $\tan\theta=\frac{5}{2}$. 14. Any point on the line $Ax+By+C=0$. 15. $(2, 1)$.
16. The new origin may be any point on the locus of the given equation.
17. $2x^2+3y^2=6$. 18. $x^2+6xy+9y^2+40\sqrt{10}\,y=0$.
19. (α) $(x^2+y^2)^2=a^2(x^2-y^2)$; (β) $x^2-y^2=a^2$;
(γ) $(x^2+y^2)^3=4k^2x^2y^2$.
20. $\rho^2=k^2\cos 2\theta$. 21. $\rho^2\cos 2\theta=49$
22. $\rho=4a$. 23. $(3+4\sqrt{3})y^2+24ay-8\sqrt{3}xy=0$.
25. $\rho\cos(\theta-\alpha)=p$.

Pages 141–142. Art. 83

1. $x^2+y^2-2xy+16x+16y+32=0$.
2. $x^2+4xy+4y^2+12x-6y-9=0$. 3. $y^2=16x$. 4. $x^2=-4ay$.
5. $x^2+4x+16y=76$. 6. $y^2-8y+12x=8$. 7. $x^2=4p(y-p)$
8. 16; $4a$; 16; 12.

Page 143. Art. 84.

1. $(3, 2)$; $(2, 2)$; 4; $x=4$; $y=2$
2. $(-1, 4)$; $(-1, \frac{49}{12})$; $\frac{1}{3}$; $12y=47$; $x+1=0$.
3. $\left(\frac{13}{48}, \frac{5}{6}\right)$; $\left(\frac{-1}{16}, \frac{5}{6}\right)$; $\frac{4}{3}$; $48x=29$; $6y=5$.

4. $(-b, a)$; $(-b+\frac{1}{16}, a)$; $\frac{1}{4}$; $x=-(b+\frac{1}{16})$; $y=a$.

5. $\left(-\frac{G}{A}, \frac{G^2-AC}{2AF}\right)$; $\left(-\frac{G}{A}, \frac{G^2-F^2-AC}{2AF}\right)$; $\frac{2F}{A}$;

$y=\frac{G^2+F^2-AC}{2AF}$; $x=-\frac{G}{A}$.

6. $\left(\frac{F^2-BC}{2BG}, -\frac{F}{B}\right)$; $\left(\frac{F^2-G^2-BC}{2BG}, -\frac{F}{B}\right)$; $\frac{2G}{B}$;

$x=\frac{F^2+G^2-BC}{2BG}$; $y=-\frac{F}{B}$.

Pages 150–151. Art. 88.

1. $7x^2-2xy+7y^2-42x-42y+135=0$.

2. $3x^2+4y^2+18x=81$.

3. $16x^2+15y^2-16y=48$.

4. $\frac{x^2}{3}+\frac{y^2}{7}=1$; $\frac{6}{\sqrt{7}}$.

5. $3x^2+5y^2=32$.

6. $\frac{x^2}{81}+\frac{y^2}{72}=1$.

7. $\frac{x^2}{45}+\frac{y^2}{81}=1$.

8. $\frac{4x^2}{81}+\frac{4y^2}{45}=1$.

9. $16x^2+25y^2=625$.

10. $\frac{x^2}{25}+\frac{y^2}{9}=1$.

11. $\frac{(x-3)^2}{16}+\frac{(y+2)^2}{9}=1$; $(3\pm\sqrt{7}, -2)$.

12. $\frac{(x-1)^2}{16}+\frac{(y-8)^2}{81}=1$; $(1, 8\pm\sqrt{65}$.

13. $\frac{33x^2}{196}+\frac{(y-7)^2}{49}=1$; $(0, 7\pm 7\sqrt{\frac{29}{33}})$.

14. $$\frac{\left(x-\frac{a-b}{2}\right)^2}{\left(\frac{a+b}{2}\right)^2}+\frac{\left(y-\frac{c-d}{2}\right)^2}{\left(\frac{c+d}{2}\right)^2}=1;$$

foci are: $\left(\frac{a-b\pm\sqrt{(a+b)^2-(c+d)^2}}{2}, \frac{c-d}{2}\right)$

if $a+b>c+d$, $\left(\frac{a-b}{2}, \frac{c-d\pm\sqrt{(c+d)^2-(a+b)^2}}{2}\right)$ if $a+b<c+d$.

15. A circle.

Pages 152–153. Art. 89.

1. $(2, 1)$; $a=3$, $b=\sqrt{3}$; $(2\pm\sqrt{6}, 1)$; $(5, 1)$, $(-1, 1)$; 2.

2. $(1, -1)$; $b=1$, $a=2$; $(1, -1\pm\sqrt{3})$; $(1, 1)$, $(1, -3)$; 1.

3. $(-2, -2)$; $b = \dfrac{7}{\sqrt{15}}$, $a = 7$; $(-2, -2 \pm \frac{7}{15}\sqrt{210})$; $(-2, -2 \pm 7)$; $\frac{14}{15}$.

4. $\left(\dfrac{-G}{A}, \dfrac{-F}{B}\right)$; $a = \sqrt{\dfrac{K}{A}}$, $b = \sqrt{\dfrac{K}{B}}$; $\left(\dfrac{-G}{A} \pm \sqrt{\dfrac{K(B-A)}{AB}}, \dfrac{-F}{B}\right)$; $\left(\dfrac{-G}{A} \pm \sqrt{\dfrac{K}{A}}, -\dfrac{F}{B}\right)$; $\dfrac{2\sqrt{KA}}{B}$, where $K = \dfrac{BG^2 + AF^2 - ABC}{AB}$.

5. $\dfrac{x^2}{9} + \dfrac{y^2}{3} = 1$; $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$; $\dfrac{x^2}{\frac{49}{15}} + \dfrac{y^2}{49} = 1$.

Pages 157–158. Art. 93.

1. $x^2 - 3y^2 - 2x + 30y - 62 = 0$.

2. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$. 3. $\dfrac{x^2}{\frac{25600}{89}} - \dfrac{y^2}{64} = 1$. 4. $2x^2 - 3y^2 = 30$.

5. $\dfrac{x^2}{a^2} - \dfrac{y^2}{3a^2} = 1$. 6. $\dfrac{(x-4)^2}{25} - \dfrac{(y-5)^2}{36} = 1$.

7. $\dfrac{(y+5)^2}{9} - \dfrac{(x+4)^2}{4} = 1$. 8. $(4 \pm \sqrt{61}, 5)$, $\frac{72}{5}$; $(-4, -5 \pm \sqrt{13})$, $\frac{8}{3}$.

9. $\dfrac{x^2}{25} - \dfrac{y^2}{36} = 1$; $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$.

10. (α) $(\pm 3, 0)$; (β) $(\pm\sqrt{19}, 0)$; (γ) $(0, \pm 5)$.

Page 159. Art. 94.

1. $\left(\dfrac{-9}{4}, -2\right)$; $a = \frac{3}{8}\sqrt{6}$, $b = \frac{3}{4}\sqrt{3}$; $\left(\dfrac{-9}{4}, -2 \pm \frac{9}{8}\sqrt{2}\right)$; $\left(\dfrac{-9}{4}, -2 \pm \frac{3}{8}\sqrt{6}\right)$; $\frac{3}{2}\sqrt{6}$.

2. $\left(5, \frac{3}{2}\right)$; $a = \dfrac{\sqrt{51}}{2}$, $b = \dfrac{\sqrt{255}}{10}$; $\left(5 \pm \frac{3}{10}\sqrt{170}, \frac{3}{2}\right)$; $\left(5 \pm \frac{1}{2}\sqrt{51}, \frac{3}{2}\right)$; $\frac{1}{5}\sqrt{51}$.

3. $(-1, 1)$; $a = \sqrt{3}$, $b = 3$; $(-1 \pm 2\sqrt{3}, 1)$; $(-1 \pm \sqrt{3}, 1)$; $6\sqrt{3}$.

4. $\left(\dfrac{-G}{A}, \dfrac{F}{B}\right)$; $a = \sqrt{\dfrac{K}{A}}$, $b = \sqrt{\dfrac{K}{B}}$; $\left(\dfrac{-G}{A} \pm \sqrt{\dfrac{K(A+B)}{AB}}, \dfrac{+F}{B}\right)$; $\left(\dfrac{-G}{A} \pm \sqrt{\dfrac{K}{A}}, \dfrac{F}{B}\right)$; $\dfrac{2\sqrt{KA}}{B}$, where $K = \dfrac{BG^2 - AF^2 - ABC}{AB}$.

5. $\dfrac{y^2}{\frac{27}{32}} - \dfrac{x^2}{\frac{27}{16}} = 1$; $\dfrac{x^2}{\frac{51}{4}} - \dfrac{y^2}{\frac{51}{20}} = 1$; $\dfrac{x^2}{3} - \dfrac{y^2}{9} = 1$.

Pages 163–164. Art. 97.

5. $k = \frac{17}{2}$ or $\frac{171}{20}$; the lines corresponding to $k = \frac{17}{2}$ are $3x + 4y + 5 = 0$, $2x + 3y + 4 = 0$, and the angle between them is $\tan^{-1}\left(\frac{1}{18}\right)$, their point of intersection is $(1, -2)$.

6. $k = -10$, or $-\frac{35}{2}$; for $k = -10$, the lines are $3x - y + 2 = 0$, $4x - 2y + 1 = 0$, the angle between them is $\tan^{-1}\left(\frac{1}{7}\right)$, their point of intersection is $\left(-\frac{3}{2}, -\frac{5}{2}\right)$.

7. $k = 1$; $y + 5 = 0$, $y - x = 0$; $(-5, -5)$; $45°$.

8. $B^2 - AC > 0$; $B^2 - AC < 0$; $B^2 = AC$; $A + C = 0$.

Page 168. Art. 98.

1. Center $\equiv (0, \sqrt{2})$; foci $\equiv (\pm\sqrt{6}, \sqrt{2} \pm \sqrt{6})$; axes: $y - x = \sqrt{2}$, $y + x = \sqrt{2}$; directrices: $x + y = \dfrac{2 \pm 3\sqrt{3}}{\sqrt{2}}$.

2. Parabola; $\theta = 30°$; vertex, referred to old axes, is at point $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$; and the latus rectum is $4\sqrt{3}$.

3. Hyperbola; $\theta = \sin^{-1}\left(\frac{2}{3}\right)$; center, referred to old axes, is at the point $\left(\dfrac{11\sqrt{5}}{18}, \dfrac{5}{9}\right)$; and $a^2 = \frac{465}{324}$, $b^2 = \frac{465}{648}$.

4. Parabola; $\theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$; vertex, referred to old axes, is at point $\left(\dfrac{-297}{50}, \dfrac{579}{200}\right)$, and the latus rectum is $\dfrac{\sqrt{5}}{50}$.

5. Hyperbola; $\theta = 22\frac{1}{2}°$; center, referred to old axes, is at the point $(-3, -1)$; and $a^2 = b^2 = 11\sqrt{2}$.

Pages 173–174. Art. 103.

1. Hyperbola; $\theta = 45°$; coördinates of center (old axes) are: $\left(\dfrac{116}{21}, \dfrac{-80}{21}\right)$; $a^2 = \frac{3158}{147}$, $b^2 = \frac{3158}{63}$.

2. Parabola; $\theta = 45°$; the new equation is:

$$\left(y - \frac{3}{4\sqrt{2}}\right)^2 = -\frac{\sqrt{2}}{4}\left(x - \frac{25\sqrt{2}}{16}\right).$$

3. Ellipse; $\theta = 0$; center (old axes) $\equiv \left(\dfrac{1}{3}, \dfrac{-1}{4}\right)$; $a^2 = \frac{35}{48}$, $b^2 = \frac{35}{72}$; foci on new y-axis.

4. Two straight lines: $3x - y + 5 = 0$, $x + 3y - 2 = 0$.

5. Two parallel lines: $3x - y + 1 = 0$, $3x - y + 2 = 0$.

6. A "point ellipse," or two imaginary straight lines through origin.

7. Circle; center (old axes) $\equiv (-\frac{11}{6}, \frac{5}{6})$; $r^2 = \frac{31}{18}$.

8. Hyperbola; center (old axes) $\equiv (-3, -1)$; $\theta = 22\frac{1}{2}°$; $a^2 = b^2 = 11\sqrt{2}$.

9. Two straight lines: $2x + y = 0$, $x - y = 3$.

10. The straight line $3x + 4y = 10$, taken twice.

15. The ellipse $107x^2 - 164xy + 63y^2 - 139x + 139y - 150 = 0$.

16. The parabola $9x^2 - 48xy + 64y^2 + 94x + 72y - 736 = 0$; pair of parallel lines $x^2 + 2xy + y^2 - 3x - 3y - 28 = 0$.

17. Pair of straight lines $x^2 + 8xy + 7y^2 - 56y - 8x = 0$.

Pages 177–178. Art. 106.

1. Yes. 2. $2y = 8x + 13$, $2y = 8x - 41$.

3. $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$; $\dfrac{y_1x}{b^2} - \dfrac{x_1y}{a^2} = x_1y_1\left(\dfrac{a^2 - b^2}{a^2b^2}\right)$.

4. $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$; $\dfrac{y_1x}{b^2} + \dfrac{x_1y}{a^2} = x_1y_1\left(\dfrac{a^2 + b^2}{a^2b^2}\right)$.

5. $x_1x = 2p(y + y_1 + 10)$; $2px + x_1y = x_1(2p + y_1)$.

6. $2(2x_1 - 3)x - 5y_1y - 6x_1 = 0$; $5y_1x + 2(2x_1 - 3)y = 3y_1(3x_1 - 2)$.

7. $(2x_1 - 3)x + 6(y_1 + 2)y = 3(x_1 - 4y_1 - 6)$; $6(y_1 + 2)x - (2x_1 - 3)y = 12x_1 + 4x_1y_1 + 3y_1$.

8. $4(x_1 + 1)x + (y_1 - 1)y = y_1 - 4x_1 - 1$. 9. $y = 1$; $x + 2 = 0$.

10. Point is $\left(-\dfrac{1}{3} + \dfrac{2}{9}\sqrt{2}, -\dfrac{1}{2} + \dfrac{\sqrt{10}}{6}\right)$ and the equations are $30y + 18\sqrt{5}x + 6\sqrt{5} - 9\sqrt{10} + 15 = 0$ and $54y - 18\sqrt{5}x - 5\sqrt{10} - 6\sqrt{5} + 27 = 0$.

11. $x + y + 4 = 0$; $y = x + 2$. 12. $\sqrt{4}x + y = 5$; $x - \sqrt{4}y = 0$.

13. $2x - 3y = 8$; $3x + 2y + 1 = 0$.

14. $y + (2 \pm \sqrt{2})x + 9 \pm 4\sqrt{2} = 0$.

Pages 181–182. Examples on Chapter VIII.

1. $y = \pm \frac{2}{5}\sqrt{10}x - 2$. 2. $4x^2 - 3y^2 + 16x = 0$, an hyperbola.

4. $(0, \sqrt{57})$, $(\sqrt{3} + \sqrt{19}, -3)$ and $(-\sqrt{3} - \sqrt{19}, -3)$; or $(0, -\sqrt{57})$ $(\sqrt{3} + \sqrt{19}, 3)$ and $(-\sqrt{3} - \sqrt{19}, 3)$.

5. $18y + 6x = 5$; $18y + 6x = 17$.

7. Hyperbola; $(-\frac{1}{2} \pm \frac{7}{2}\sqrt{2}, 0)$; $x = -\frac{1}{2} \pm \frac{7}{4}\sqrt{2}$; $(-\frac{1}{2}, 0)$; $\frac{7}{2}, \frac{7}{2}$; 7.

8. This equation may be written: $\frac{(x-1)^2}{\frac{8}{3}} - \frac{(y+1)^2}{4} = 1$, which shows that it is an hyperbola; that the center is at $(1, -1)$; that the transverse axis is $\frac{4}{3}\sqrt{6}$ and is parallel to the x-axis, etc.

9. This equation may be written: $(x-2a)^2 = 4p\left(y + b + \frac{3a^2}{4p}\right)$, — a parabola.

10. This equation may be written: $\frac{(x+4)^2}{3} + \frac{(y+2)^2}{2} = 1$, — an ellipse.

11. This equation may be written: $-\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$, — an hyperbola.

12. This equation may be written: $\frac{(x-a)^2}{2(6a^2+1)} + \frac{(y+a)^2}{6a^2+1} = 1$, — an ellipse.

15. $\rho = \frac{27}{1 - 3\cos\theta}$.

Pages 186–188. Art. 113.

3. $y = \frac{1}{3}\sqrt{3}x + \sqrt{3}p$, and $y = -\frac{1}{3}\sqrt{3}x - \sqrt{3}p$; 1 : 4.

4. $12y = 9x + 16p$; $\left(\frac{16p}{9}, \frac{8p}{3}\right)$. **5.** $y = (\frac{1}{2} \pm \frac{1}{4}\sqrt{14})x + 1 \mp \frac{1}{2}\sqrt{14}$.

6. $x \pm y = 5$; the directrix and the x-axis.

7. $x + 2y + 10 = 0$; $2x - y = 30$. **8.** $8x - 4y - 9 = 0$.

9. $4x - 2y + p = 0$; $\sqrt{3}y = x + 3p$. **10.** $x + y + p = 0$.

11. $y^2 = p(x-p)$. **13.** 6, $6\sqrt{2}$. **14.** $y^2 = -\frac{4ac}{b^2}x$.

Pages 189–191. Art. 113.

8. The value of $y + 3 = mx$, where m are the roots of the equation $5m^3 + 10m + 6 = 0$.

Pages 191–192. Art. 115.

1. $3y = 32$. **2.** $3y - 4x = k$. **3.** $3y + 4 = 0$. **4.** $x + 2 = 0$.

Pages 192–194. Review Exercises.

1. $y^2 = x$. **2.** $y^2 - 6y = 8x - 25$. **3.** $(y+1)^2 = 8(x+2)$.

4. $x^2 - 2x + y + 2 = 0$. **5.** $(1, -1)$; $x = 1$. **6.** $y^2 = 4p(x-p)$.

7. $y^2 = 4p(x+p)$. **8.** $y^2 = \frac{64}{3}x$. **9.** $y^2 = 2x$, $y^2 = -18(x-5)$.

10. $y = x$; $x + y + p = 0$. **11.** $90°$; $\tan^{-1}(\frac{3}{4})$. **12.** $4p^2$.

13. $y^2 = px$. 14. $y^2 = 2\,px$. 17. $kx = p$. 18. $y = kp$.

19. $y^2 = 4\,px + p^2k^2$.

20. A parabola whose focus is the given point and whose directrix is the given line.

22. $ky^2 = p(1+k)^2\,x$, if k is the given constant ratio.

23. $y + 2\,x + 5 = 0$.

24. A parabola whose focus is the center of the given circle, and whose directrix is a line parallel to the given line and at a distance from it equal to the radius of the given circle.

25. $y^2 = 4\,p(x - 4\,p)$.

26. $\left(x_1 + 4\,p + \frac{4\,p^2}{x_1},\ -y_1 - \frac{8\,p^2}{y_1}\right);\ \frac{4\sqrt{p\cdot(p+x)^3}}{x_1}$.

Pages 197-198. Art. 118.

4. $3x - 4y = \pm 4\sqrt{13}$; $3\,x - 4\,y = \pm\frac{72}{\sqrt{73}}$. 5. $y = 4$ or $2\,y + 3\,x = 17$.

7. Through the points for which $x = \frac{\pm a^2}{\sqrt{a^2+b^2}}$.

8. $\sqrt{7}\,x + 4\,y = 16$; $4\,x - \sqrt{7}\,y = \frac{7}{4}\sqrt{7}$; $\frac{16}{7}\sqrt{7}$; $\frac{7}{16}\sqrt{7}$.

9. The points for which $x = \pm\frac{1}{14}\sqrt{105}$. 10. $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

11. $x + y = 5$; $y = 3$. 12. $y = -\frac{1}{2}\,x \pm 3$, $y = -2\,x \pm \sqrt{39}$.

13. $\frac{b^4}{a^2(1 - e^2\cos^2\theta)}$.

14. The points for which $x = \frac{\pm a^2}{\sqrt{a^2+b^2}}$; the same.

Pages 206-207. Art. 124.

1. $27\,y + 20\,x = 0$. 2. $5\,y = 3\,x$.

4. (α) $a^3y + b^3x = 0$; (β) $ay + bx = 0$.

6. $a' = b' = \sqrt{\frac{29}{2}}$. 7. $\tan^{-1}(-2)$; $2\sqrt{5}\,y = x$.

8. $(\frac{3}{2}\sqrt{2},\ \sqrt{2})$; $3\,y + 2\,x = 0$. 9. $\sqrt{26}$.

10. $(\pm 3,\ \pm\sqrt{3})$; $\tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right) = 30°, 150°$.

Pages 207-209. Review Exercises.

1. $(\pm 1, 0)$; $x = \pm 4$; $\frac{1}{2}$. 2. $\pi m^{\frac{3}{2}} n^{\frac{3}{2}}$.

4. $(x^2 + y^2)^2 = 4\,x^2 + y^2$.

6. $364x^2 + 400y^2 - 1092x = 8281$. 7. 4. 8. $9;\ \frac{9}{2};\ \frac{9}{4}\pi\sqrt{3}$.

9. If the generating point divides the line in the ratio $a : b$, the equation of the locus is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

10. $\dfrac{x^2}{\frac{a^2}{4}} + \dfrac{\left(y - \frac{b}{2}\right)^2}{\frac{b^2}{4}} = 1$. 13. $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

15. If the base coincide with the x-axis, its middle point at the origin, the equation is $b^2x^2 + c^2y^2 = a^2b^2$ 17 $\sqrt{\frac{h^2}{a^2} + \frac{k^2}{b^2}}$.

19. $(y^2 - x^2 + a^2 - b^2)\tan 2\alpha + 2xy = 0$. 20. $2xy = c(x^2 - a^2)$.

Pages 214–215. Art. 129.

3. $\frac{x^2}{16} - \frac{y^2}{9} = 1;\ 8, 6;\ (\pm 5, 0);\ \frac{9}{2}$.

4. $9x - 8y + 11 = 0;\ 8x + 9y + 42 = 0;\ -\frac{16}{9};\ -\frac{13}{4}$.

5. $\left(\pm \frac{a^2}{\sqrt{a^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 - b^2}}\right)$. 6. $\frac{9}{2};\ \frac{17}{2}$. 7. ± 6.

8. None. 9. $y = \pm \frac{2}{3}\sqrt{42}\,x + 5$. 11. $3x^2 - y^2 = 3a^2$.

14. $\frac{1}{2}\sqrt{5};\ a$. 15. $y = \pm\left(\frac{x}{\sqrt{3}} - 1\right);\ 2\sqrt{3}(3 - \sqrt{5})$; there are three other solutions

16. $\frac{4}{3}\sqrt{5};\ 3\sqrt{5};\ \frac{3}{2};\ 6$.

Pages 220–221. Art. 132

1. $4y = \pm 3x;\ 2\tan^{-1}\frac{3}{4} = \tan^{-1}\frac{24}{7}$. 2. $8x^2 - y^2 = 8a^2;\ y = \pm 2\sqrt{2}\,x$.

5. $x^2 - y^2 + 10x + 4y = 11$. 6. $x^2 - y^2 + 10x + 4y + 53 = 0$.

7. $x - 2y = 5,\ x + 2y + 3 = 0;\ x^2 - 4y^2 - 2x - 16y - 11 = 0$.

8 $7y + x = 0,\ y + 7x = 0$. 9. $4x^2 - 9y^2 + 8x - 18y + 31 = 0$

Pages 223–224. Art. 134.

1. $5y + x = 0;\ y + 4x = 0$. 2. $y = x;\ 5y = 4x$. 3. $16y = 45x$.

4. $3y - 8x = 52$.

Pages 224–226. Review Examples.

1. $\frac{x^2}{3} - \frac{y^2}{9} = 1.$ **2.** $15\,y = 16\,x.$ **4.** $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$, if the middle point of the base is the origin and the base of the triangle on the x-axis.

5. $2\,xy = k(c^2 - x^2)$, if k is the constant and axes chosen as in Ex. 4.

6. $(x-5)(y-4) = 0;\ xy - 5\,y - 4\,x + 40 = 0.$

11. $\left(\mp\frac{3\,a\sqrt{6}}{8},\ \pm\frac{a\sqrt{6}}{8}\right),\ \left(\pm\frac{a\sqrt{6}}{6},\ \pm\frac{a\sqrt{6}}{2}\right).$

15. $y = \pm 2\,x \pm \frac{1}{2}\sqrt{15}.$ **16.** $6\,y - 5\,x = 7.$ **19.** $y_1x + x_1y = 8.$

22. $\sqrt{b^2 + ab}\,x \pm \sqrt{a^2 - ab}\,y \pm \sqrt{ab(a^2 + b^2)} = 0.$

23. $32\,y = (\sqrt{3281} \pm 41)x.$ **25.** $x + y = \sqrt{a^2 + b^2}$

26. $2\,x + 3\,y + 4 = 0;\ 3\,y^2 + 2\,xy + 4\,y + 9 = 0.$

Pages 232–233. Examples on Chapter X.

2. $\frac{-3}{\sqrt{14}},\ \frac{-2}{\sqrt{14}},\ \frac{1}{\sqrt{14}}.$ **3.** $\cos\theta = \frac{4}{45}\sqrt{30}.$

5. $\alpha = \beta = \gamma = \cos^{-1}\frac{1}{\sqrt{3}}.$ **6.** $90°.$ **8.** $\sqrt{110}.$

9. $\rho_1 = \sqrt{14},\ \cos\alpha_1 = -\frac{1}{\sqrt{14}},\ \cos\beta_1 = -\frac{2}{\sqrt{14}},\ \cos\gamma_1 = -\frac{3}{\sqrt{14}}.$

$\rho_2 = \sqrt{2},\ \cos\alpha_2 = 0,\ \cos\beta_2 = -\frac{1}{\sqrt{2}},\ \cos\gamma_2 = -\frac{1}{\sqrt{2}}.$

$\rho_3 = \sqrt{a^2 + b^2 + c^2},\ \cos\alpha_3 = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ \cos\beta_3 = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$

$\cos\gamma_3 = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$

11. Internally: $(\frac{3}{6}, \frac{4}{6}, \frac{5}{6})$; externally: $\left(\frac{-3}{4}, -1, \frac{-5}{4}\right).$

Pages 242–243. Examples on Chapter XI.

1. Two coincident planes parallel to the yz-plane and at the distance $+\frac{1}{5}$ from it.

2. The yz-plane, and a plane parallel to it, and at a distance -4 from it.

3. Two planes parallel to the z-axis, and intersecting the xy-plane in the lines $x - y = 0$ and $x - y + 3 = 0.$

4. Two planes intersecting in the z-axis, and intersecting the xy-plane in the lines $y = \frac{b \pm \sqrt{b^2 - 4\,ac}}{2\,c}\,x.$

5. Three planes parallel to the yz-plane, and at distances -1, $+2$, -2 from it.

6. A parabolic cylinder with generators parallel to the z-axis.

7. A circle parallel to the yz-plane, with center on the x-axis and radius 3.

8. A pair of lines respectively parallel to $2x = \pm 3y$.

9. The projection of this curve upon the xz-plane is the hyperbola $3x^2 - z^2 + 5 = 0$, and its projection on the yz-plane is the ellipse $3y^2 + 4z^2 = 32$.

10. For $z = 5$, the point $(0, 0, 5)$; for $z = -5$ it is a circle parallel to the xy-plane, and whose equation is $9x^2 + 9y^2 = 100$.

11. Solved like No. 9. 12. The point $(0, 0)$; $x = \pm 2z$; $y = \pm 2z$.

13. $x^2 - 3y^2 = 6$; $x^2 + 2z^2 = 6$; $3y^2 - 2z^2 + 6 = 0$.

14. $x^2 + y^2 = 5z + 3$. 15. $x^2 + z^2 = (y - 10)^2$.

16. $\dfrac{x^2 + y^2}{3} + \dfrac{z^2}{2} = 1$. 17. $\dfrac{(x-1)^2}{7} + \dfrac{y^2}{7} + \dfrac{(z-1)^2}{9} = 1$.

18. $(y^2 + z^2)x + 4 = 0$. 19. $16y^2 - 9x^2 - 9z^2 = 144$.

20. $36x^2 + 36y^2 + 36z^2 - 36x - 24y + 5 = 0$.

Pages 251–252. Examples on Chapter XII.

1. $x - 2y = 3$, $x - z = 2$. 2. $7x - 5y - 5z = 0$.

3. $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$.

4. $x - 2y = 3$ (on xy-plane), $x - z = 2$ (on xz-plane), $z - 2y = 1$ (on yz-plane); it pierces the xy-plane at $\left(2, \dfrac{-1}{2}, 0\right)$, the yz-plane at $\left(0, \dfrac{-3}{2}, -2\right)$, and the xz-plane at $(3, 0, 1)$.

5. $\dfrac{x+1}{1} = \dfrac{y-2}{3} = \dfrac{z+3}{3}$. 6. $\dfrac{x}{\frac{7}{2}} + \dfrac{y}{\frac{7}{3}} + \dfrac{z}{\frac{7}{10}} = 1$;

$$\frac{2x}{\sqrt{113}} + \frac{3y}{\sqrt{113}} + \frac{10z}{\sqrt{113}} = \frac{7}{\sqrt{113}}.$$

7. $\dfrac{x}{-\frac{10}{c}} + \dfrac{y}{\frac{10}{a}} + \dfrac{z}{\frac{10}{b}} = 1$; $-\dfrac{-cx}{\sqrt{a^2+b^2+c^2}} + \dfrac{ay}{\sqrt{a^2+b^2+c^2}} + \dfrac{bz}{\sqrt{a^2+b^2+c^2}}$

$$= \frac{10}{\sqrt{a^2+b^2+c^2}}.$$

8. $\frac{x-5}{1}=\frac{y}{3}=\frac{z-3}{-7}$; $\frac{1}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{-7}{\sqrt{59}}$. 9. $x\cos\alpha+y\cos\beta=p$; $z=p$

11. $3x-4y+10z=0$; $3x-4y+10z=28$.

12. $11x-7y-2z=22$. 13. $-\frac{1}{3}\sqrt{30}$; $-\frac{31}{30}\sqrt{30}$; yes.

14. $\cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)$ for the yz-plane, $\cos^{-1}\left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)$ for the xz-plane, and $\cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$ for the xy-plane; $\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)$, $\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)$, and $\sin^{-1}\left(\frac{c}{\sqrt{a^2+b_2+c^2}}\right)$ respectively for the x-, y-, and z-axes.

15. $13x+11y+z=25$. 16. $x+y+z=d\sqrt{3}$.

17. $\frac{x+4}{4}=\frac{y+5}{-5}=\frac{z+6}{1}$.

18. $2x-4z=3$, $6x+4y+1=0$, $2y+6z+5=0$.

Page 264. Examples on Chapter XIII.

5. $(x_1-a)(x-a)+(y_1-b)(y-b)+(z_1-c)(z-c)=r^2$ is the tangent plane at (x_1, y_1, z_1). 7. $\frac{x^2}{25}-\frac{y^2}{4}+\frac{z^2}{9}=0$.

8. $$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2+\left(z-\frac{z_1+z_2}{2}\right)^2 = \frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{4}$$

10. $\frac{x^2}{64}+\frac{y^2}{4}+\frac{z^2}{64}=1$.

Polar Coordinates

The intercepts on the polar axis are obtained by putting $\theta = 0$ and $\theta = \pi$ and solving for ρ. Other values of θ may make ρ equal to 0 and give a pt. on the polar axis (the pole itself)

2. The curve is symetrical with respect to the pole if when $-\rho$ is substituted for ρ only the form of eqn. is changed

3. The curve is symetrical with respect to the polar axis if when $-\theta$ is subs. for θ only the form of the eqn is changed.

4. The directions from the pole in which the curve recedes to infinity, if any, are found

by obtaining those values of θ for which C becomes infinite

5. The method of finding the values of θ which must be excluded depends on the given eqn.